

GEOMETRY OF LAGRANGIAN SELF-SHRINKING TORI AND APPLICATIONS TO THE PIECEWISE LAGRANGIAN MEAN CURVATURE FLOW

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ABSTRACT. We study geometric properties of the Lagrangian self-shrinking tori in \mathbb{R}^4 . When the area is bounded above uniformly, we prove that the entropy for the Lagrangian self-shrinking tori can only take finitely many values; this is done by deriving a Łojasiewicz-Simon type gradient inequality for the branched conformal self-shrinking tori and then combining with the compactness theorem in [5]. When the area bound is small, we show that any Lagrangian self-shrinking torus in \mathbb{R}^4 with small area is embedded with uniform curvature estimates, and the space of such tori is compact.

Using the finiteness of entropy values, we construct a piecewise Lagrangian mean curvature flow for Lagrangian immersed tori in \mathbb{R}^4 , along which the Lagrangian condition is preserved, area is decreasing, and the type I singularities that are compact with a fixed area upper bound can be perturbed away in finite steps. This is a Lagrangian version of the construction for embedded surfaces in \mathbb{R}^3 in [6].

1. INTRODUCTION

One of the major challenging problems in the study of Lagrangian mean curvature flow is to formulate a weak version of the mean curvature flow that preserves the Lagrangian condition and goes beyond singular time, as the well-known weak forms of mean curvature flow such as the Brakke flow or the level set approach do not work well in the Lagrangian setting.

It is known that the rescaled mean curvature flow (MCF) at a finite time singularity converges to a self-shrinking solution, the so-called *self-shrinker*; the convergence may be weak if the singularity is not of type I (cf. [14], [15], [32]). As local models of singularities, self-shrinkers are important. In [6], Colding and Minicozzi introduced an entropy functional (see (2.9)) of a hypersurface (cf. [22]) and showed that the sphere and the cylinders are the only entropy stable self-shrinking hypersurfaces. Using this and a compactness theorem [7] on the space of embedded self-shrinking surfaces in \mathbb{R}^3 , they constructed in [6] a piecewise MCF for embedded surfaces in \mathbb{R}^3 (under some assumptions), such that if a uniform diameter estimate holds then the flow shrinks to a round point.

When the initial immersion is Lagrangian, a basic fact proved by Smoczyk [28] is that the MCF preserves the Lagrangian condition. In [18], [20] the authors studied the Lagrangian entropy stability of Lagrangian self-shrinking immersions and obtained entropy instability results. In particular, Li and Zhang showed in [20] that if $F : M^n \rightarrow \mathbb{R}^{2n}$ is a closed orientable Lagrangian self-shrinker and the first Betti number of M is

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greater than 1 then F is Lagrangian entropy unstable¹. Since there is no simply connected closed Lagrangian self-shrinker (see [29], and [6] for branched immersions in dimension 2), all closed orientable Lagrangian self-shrinkers in \mathbb{R}^4 are Lagrangian entropy unstable.

In [5], we used the harmonic map theory to show that if $\{F_n : (\Sigma, h_n) \rightarrow \mathbb{R}^4\}$ is a sequence of compact Lagrangian self-shrinking immersions with uniform area bound and that the conformal structures h_n do not degenerate, then a subsequence converges to a *branched conformal* Lagrangian self-shrinker $F : (\Sigma, h) \rightarrow \mathbb{R}^4$. Furthermore, the assumption on the conformal structure can be dropped if Σ is topologically a torus \mathbb{T} . This provides a compactification of the space of compact Lagrangian self-shrinking immersions in \mathbb{R}^4 . A key ingredient in proving the compactness result is the rigidity established in [6]: there are no *branched* Lagrangian self-shrinking 2-spheres. This prevents development of the bubbles in Sacks-Uhlenbeck's compactness procedure [30], [23], hence yields strong convergence for the harmonic mappings.

The goal of this paper is twofold. First, we establish geometric properties of branched conformal Lagrangian self-shrinking tori. The discussion is divided into two cases: for small area bound we prove embeddedness and curvature estimates, and for arbitrary area bound we show finite discreteness of the values of the entropy. To achieve the latter, we derive a Łojasiewicz-Simon type gradient inequality for the energy operator \mathcal{E} naturally defined on the total space of $C^{2,\alpha}$ -mappings and the moduli space of the conformal structures on the torus. This should have other applications. Second, the Lagrangian entropy instability, compactness for the space of Lagrangian self-shrinking tori and the finiteness of the value distribution of the entropy in Theorem 1.1 below together lead us to define a piecewise Lagrangian MCF for a Lagrangian immersed torus in \mathbb{R}^4 which preserves the Lagrangian condition and the Maslov class, decreases area and avoids compact type I singularities with an arbitrarily given area upper bound in a finite number of steps.

We now state our results on the Lagrangian self-shrinking tori.

Definition 1.1. Let Λ be a positive number. Let \mathfrak{X}_Λ be the space of branched conformally immersed Lagrangian self-shrinking tori with area less than or equal to Λ .

When the area upper bound Λ is not small (as in Theorem 1.2), it is not known whether any branched conformal Lagrangian self-shrinking torus with nonempty branch locus exists or not. The possible existence of branch points of elements in \mathfrak{X}_Λ is a serious obstacle for applications to Lagrangian MCF as one would hope to perturb the branched Lagrangian surface to a nearby Lagrangian immersion, but such resolution of singularity in the Lagrangian setting, even in dimension two, is not available. Note that it is in general difficult to study nearby branched immersions by deforming them along the normal vector fields. In particular, it is hard to study stability problem of branched Lagrangian self-shrinking immersions as in [6], [20], and Weinstein's Lagrangian neighbourhood theorem [31] does not apply to the branched case. In view of all these

¹More precisely, in [18], [20], the authors study the Lagrangian \mathcal{F} -stability of a Lagrangian immersion. The equivalence of \mathcal{F} -stability and entropy stability is proved in [6] for the hypersurface case and can be generalized to higher codimensional cases. See section 2 and [1].

and the special feature of the embedded graphic representation of a surface near a self-shrinker in the codimension one case, the idea of the piecewise MCF introduced in [6] is not directly applicable to the Lagrangian case in \mathbb{R}^4 , even with the compactness theorems in [5].

In order to construct a piecewise Lagrangian MCF for torus, we observe in this paper that one can bypass the issue of branchedness of a limiting surface in \mathfrak{X}_Λ by controlling the entropy values $\lambda(F)$ attained by the self-shrinkers, where for a branched immersion $F : \mathbb{T} \rightarrow \mathbb{R}^4$ its entropy is defined by

$$\lambda(F) = \sup_{x_0 \in \mathbb{R}^4, t_0 > 0} \frac{1}{4\pi t_0} \int_{\mathbb{T}} e^{-\frac{|F(x) - x_0|^2}{4t_0}} d\mu_F.$$

The theorem below is a crucial ingredient in our construction of piecewise Lagrangian MCF for torus, but it is also interesting in its own right: it is equivalent to that in the induced metric from $G = e^{-\frac{|x|^2}{4}} \delta_{ij}$ on \mathbb{R}^4 the areas of branched Lagrangian self-shrinking tori in \mathfrak{X}_Λ can only take a finite number (depending on Λ) of values for any given Λ .

Theorem 1.1. *Let $\lambda : \mathfrak{X}_\Lambda \rightarrow [0, \infty)$ be the entropy function which sends F to its entropy $\lambda(F)$. Then the image of λ is finite for any given Λ .*

To prove Theorem 1.1, we derive a Łojasiewicz-Simon gradient inequality for branched conformal self-shrinking 2-dimensional tori. The celebrated Łojasiewicz-Simon gradient inequality is proved in [26] with important applications to the harmonic map flow and the minimal cones. Since the pioneering work [26], the inequality and its variation has wide applications in geometric problems. For MCF, Schulze [25] used the inequality to prove a uniqueness result for compact embedded singularity of tangent flow. Colding and Minicozzi [8] derived Łojasiewicz-Simon gradient inequalities in a noncompact setting and settled the uniqueness problem for all generic singularities of mean convex MCF at all singularities.

The classical Łojasiewicz-Simon gradient inequality is established for real analytic functionals over a compact manifold whose Euler-Lagrange operator is elliptic and of order 2. In our case, we are concerned with the entropy functional λ , which is, at a self-shrinker, just the area of the shrinker in (\mathbb{R}^4, G) up to a universal constant. However, in our situation, the self-shrinkers might be branched and the Euler-Lagrange operator of the area functional fails to be elliptic at the branch locus, so Simon's infinite dimensional version of the Łojasiewicz inequality in [26] is not directly applicable. To overcome the difficulty, we consider the real analytic energy functional \mathcal{E} defined on the mapping space $C^{2,\alpha}(\mathbb{T}, \mathbb{R}^4)$ together with the Teichmüller space of \mathbb{T} , and continue to view self-shrinkers as branched minimal immersions in (\mathbb{R}^4, G) [2]. The functional \mathcal{E} has been extensively used in minimal surface theory, especially, in showing existence of minimal surfaces. A critical point of \mathcal{E} corresponds to a branched conformal self-shrinking torus. Since the space of conformal structures on a torus is two dimensional, the ellipticity of the L^2 -gradient of \mathcal{E} at a critical point of \mathcal{E} for each *fixed* conformal structure enables us to show that the second order derivative \mathcal{L} of \mathcal{E} at the critical point is a Fredholm operator of index zero, which is sufficient to derive the desired gradient inequality. Theorem 1.1 is then a direct consequence of the gradient inequality and the compactness results in [5].

If $\Lambda < 32\pi$, the Willmore functional of a self-shrinker with area upper bound Λ is less than 8π ; a classical theorem of Li and Yau [19] then asserts that all compact Lagrangian self-shrinking tori must be embedded and without branched point. Using recent results of Lamm-Schätzle in [17] and the compactness results in [5], we show that the upper bound can be pushed beyond Li-Yau's estimate:

Theorem 1.2. *There are positive numbers ϵ_0, ϵ_1 and C_0 , where $\epsilon_1 \leq \epsilon_0$, so that*

- (1) *(Compactness) The space $\mathfrak{X}_{32\pi+\epsilon}$ is compact for all $0 < \epsilon \leq \epsilon_0$,*
- (2) *(No Branch Points) All elements in $\mathfrak{X}_{32\pi+\epsilon_0}$ are immersed, and all elements in $\mathfrak{X}_{32\pi+\epsilon_1}$ are embedded.*
- (3) *(Curvature Estimates) If $F \in \mathfrak{X}_{32\pi+\epsilon_0}$, then the second fundamental form of F is bounded by C_0 .*

Lastly, we apply Theorem 1.1 to construct a piecewise Lagrangian MCF for Lagrangian immersed torus $F : \mathbb{T} \rightarrow \mathbb{R}^4$ (see Definition 5.1). In fact, we only need the result for compactified space of immersed Lagrangian self-shrinking tori. We show that all type I singularities with an arbitrarily given area upper bound can be perturbed in finitely many steps, where a smooth Lagrangian MCF for torus restarts at each step, such that the same kind of singularities will not appear in the last step. We remark that the perturbation can be made arbitrarily small while fixing the number of perturbation performed. Note that, in the special case of small area, Theorem 1.2 is sufficient since the existence of a nearby Lagrangian immersion of the torus around a limiting surface in \mathfrak{X}_Λ (now immersed) follows from the Lagrangian neighbourhood theorem.

Our main result on Lagrangian mean curvature flow in a weak form is

Theorem 1.3. *Let $F : \mathbb{T} \rightarrow \mathbb{R}^4$ be an immersed Lagrangian torus and let $\Lambda, \delta > 0$ be given constants. Then there exists a piecewise Lagrangian MCF $\{F_t^i : i = 0, 1, \dots, k-1\}$ with initial condition F , where $k \leq |\lambda(\mathfrak{X}_\Lambda)| < \infty$, such that the singularity at time t_k is not a type I singularity modelled by a compact self-shrinker with area less than or equal to Λ . Moreover, the Maslov class of each immersion is invariant along the flow.*

Under an additional assumption, we prove a similar result in Theorem 5.1 for the case of genus larger than one.

The organization of this paper is as follows. Some background is provided in section 2. In section 3, we provide with proof necessary results in generalizing the compactness theorem in [5] from immersions to branched immersions. The compactness result is stated in Theorem 3.1. We then prove Theorem 1.2. In section 4, we derive a Łojasiewicz-Simon gradient inequality for branched conformal self-shrinking torus and prove Theorem 1.1. In the last section we prove Theorem 1.3 and a result in the higher genus case.

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2. BACKGROUND

2.1. Mean curvature flow and the self-shrinkers. A family of immersions $F_t : \Sigma \rightarrow \mathbb{R}^N$ from an n -dimensional manifold Σ to the Euclidean space is said to satisfy the mean curvature flow (MCF) if

$$(2.1) \quad \frac{\partial F_t}{\partial t} = \vec{H},$$

where \vec{H} is the mean curvature vector given by

$$\vec{H} = \text{tr} A, \quad A(X, Y) = (D_X Y)^\perp,$$

where A is the second fundamental form of the immersion and \perp denotes the normal component of a vector. An immersion is called self-shrinking (or a self-shrinker) if it satisfies

$$(2.2) \quad \vec{H} = -\frac{1}{2}F^\perp.$$

If F is self-shrinking, then up to a family of diffeomorphisms, the family of immersions

$$\{\sqrt{-t}F : t \in [-1, 0)\}$$

solves the MCF. The self-shrinkers model the singularity of MCF (cf. [14, 15, 32]).

An immersion $F : \Sigma \rightarrow \mathbb{R}^{2n}$ of an n -dimensional manifold Σ is called Lagrangian if $F^*\omega = 0$, where

$$(2.3) \quad \omega = \sum_{i=1}^n dx^i \wedge dy^i$$

is the standard symplectic form on \mathbb{R}^{2n} . If $F_t : \Sigma \times [0, T) \rightarrow \mathbb{R}^{2n}$ is the MCF starting from a Lagrangian immersion F_0 , then F_t is Lagrangian for all $t \in [0, T)$ [28].

When Σ is a surface, a branched conformal immersion $F : \Sigma \rightarrow \mathbb{R}^N$ is called a self-shrinker if (2.2) is satisfied on $\Sigma \setminus B$, where B is the set of branch points. A branched conformal immersion $F : \Sigma \rightarrow \mathbb{R}^4$ is called Lagrangian if $F^*\omega = 0$, where ω is as in (2.3) (see [5], section 3).

2.2. Lagrangian \mathcal{F} -stability and Lagrangian entropy stability. The entropy λ and \mathcal{F} -stability are introduced in [6] for an embedded self-shrinking hypersurfaces and are later carried over in [1], [18], [20] for all codimensions. The Lagrangian case is discussed in [18], [20] and the definition of Lagrangian \mathcal{F} -stability is introduced therein. We start with recalling the definitions of the \mathcal{F} and λ functionals and the related stability. When we consider Lagrangian immersions, we will assume $N = 2n$.

Definition 2.1. Let $(x_0, t_0) \in \mathbb{R}^N \times \mathbb{R}_{>0}$. The \mathcal{F} -functional of an immersion $F : \Sigma^n \rightarrow \mathbb{R}^N$ is given by

$$(2.4) \quad \mathcal{F}_{x_0, t_0}(F) = (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|F(x) - x_0|^2}{4t_0}} d\mu_F.$$

The \mathcal{F} -functional characterizes the self-shrinkers as follows: $F : \Sigma \rightarrow \mathbb{R}^N$ is a self-shrinker if and only if

$$\left. \frac{d}{ds} \mathcal{F}_{x_s, t_s}(F_s) \right|_{s=0} = 0$$

for all variations (x_s, t_s, F_s) such that $(x_0, t_0, F_0) = (0, 1, F)$.

We recall that a normal vector field X along a Lagrangian immersion is called a Lagrangian variation if

$$(2.5) \quad d(\iota_X \omega) = 0.$$

Definition 2.2. A self-shrinker F is called (Lagrangian) \mathcal{F} -stable if for all (Lagrangian) variations F_s , there is a variation (x_s, t_s) so that

$$\left. \frac{d^2}{ds^2} \mathcal{F}_{x_s, t_s}(F_s) \right|_{s=0} \geq 0.$$

In [20], Li and Zhang calculated the second variation of the \mathcal{F} -functional of a Lagrangian immersion with respect to the Lagrangian variations. They proved:

Theorem 2.1. *Let Σ be a compact orientable n -dimensional manifold whose first Betti number is greater than 1. If $F : \Sigma \rightarrow \mathbb{R}^{2n}$ is a Lagrangian self-shrinker, then F is Lagrangian \mathcal{F} -unstable.*

When $F : \Sigma \rightarrow \mathbb{R}^{2n}$ is a Lagrangian immersion, let $F_s : \Sigma \rightarrow \mathbb{R}^{2n}$ be a normal variation of F such that each F_s is a Lagrangian immersion. In this case, the normal variational vector field $X = \left. \frac{d}{ds} \right|_{s=0} F_s$ can be identified with a closed 1-form on Σ by $X \mapsto -\iota_X \omega$. The converse is also true as seen in the following elementary lemma.

Recall that if α is a 1-form on a Riemannian manifold (Σ, g) then α^\sharp is the vector field on Σ uniquely determined by

$$(2.6) \quad g(\alpha^\sharp, Y) = \alpha(Y), \quad \forall Y \in T\Sigma.$$

Lemma 2.1. *Let $F : \Sigma^n \rightarrow \mathbb{R}^{2n}$ be a Lagrangian immersion and let α be a closed 1-form on Σ . Then there is a family of Lagrangian immersions $F_s : \Sigma \rightarrow \mathbb{R}^{2n}$ so that $F_0 = F$ and*

$$(2.7) \quad \left. \frac{d}{ds} \right|_{s=0} F_t = J\alpha^\sharp,$$

where J is the standard complex structure on \mathbb{R}^{2n} .

Proof. Let $\pi : N\Sigma \rightarrow \Sigma$ be the normal bundle of the immersion F . Then the mapping

$$\tilde{F}(x, v) = F(x) + v$$

is a local diffeomorphism from a tubular neighbourhood U of the zero section of $N\Sigma$ onto its image in \mathbb{R}^{2n} .

Since α is a closed 1-form on Σ , $\beta = (\pi|_U)^* \alpha$ is a closed 1-form on U , and β sends the normal vectors v to zero. The pullback 2-form $\omega_0 = \tilde{F}^* \omega$ on U is closed as ω is closed

and it is non-degenerate as \tilde{F} is a locally diffeomorphic and ω is non-degenerate. Let X be the vector field on U dual to β with respect to ω_0 , that is,

$$(2.8) \quad \beta(Y) = -\omega_0(X, Y)$$

for all vector fields Y on U . Let ϕ_s with $s \in (-\epsilon, \epsilon)$ be the one parameter group of diffeomorphisms on U generated by X . Then $F_s := \tilde{F} \circ \phi_s|_\Sigma : \Sigma \rightarrow \mathbb{R}^{2n}$ is a family of Lagrangian immersions in \mathbb{R}^{2n} and $F_0 = \tilde{F} \circ \phi_0|_\Sigma = \tilde{F}|_\Sigma = F$.

It remains to verify (2.7). By the definition of \tilde{F} , its differential \tilde{F}_* maps the tangent vectors to the zero section $(\Sigma, 0)$ at the point $(x, 0) \in U$ to the tangent vectors to the image surface $F(\Sigma)$ at the point $F(x) \in \mathbb{R}^{2n}$ and it maps the normal vectors to the normal vectors by the identity map at the corresponding points. We need to check $X = J\alpha^\sharp$. Let Y_1, Y_2 be arbitrary tangent vectors to the zero section $(\Sigma, 0)$ at a point $(x, 0)$. Since JY_2 is normal to Σ as Σ is Lagrangian and $\omega(X, Y) = \langle JX, Y \rangle$, we have

$$\begin{aligned} \alpha(\pi_* Y_1) &= \beta(Y_1 + JY_2) \\ &= -\omega_0(X, Y_1 + JY_2) \\ &= -\omega(\tilde{F}_* X, \tilde{F}_* Y_1 + \tilde{F}_* JY_2) \\ &= -\omega(\tilde{F}_* X, \tilde{F}_* Y_1 + JY_2) \\ &= -\langle J\tilde{F}_* X, \tilde{F}_* Y_1 + JY_2 \rangle \\ &= -\langle J\tilde{F}_* X, \tilde{F}_* Y_1 \rangle - \langle \tilde{F}_* X, Y_2 \rangle \end{aligned}$$

As \tilde{F} is locally diffeomorphic, X is normal to the zero section because Y_2 is arbitrary. Then it follows from the arbitrariness of Y_1 that $-JX = \alpha^\sharp$, by dropping the notion \tilde{F}_* . This is the same as $X = J\alpha^\sharp$. \square

The entropy of a hypersurface is defined in [6, 22]. The definition for an immersion in any codimension is the same.

Definition 2.3. The entropy of an immersion $F : \Sigma \rightarrow \mathbb{R}^N$ is defined as

$$(2.9) \quad \lambda(F) = \sup_{x_0, t_0} \mathcal{F}_{x_0, t_0}(F).$$

It is clear that $\lambda(F)$ is invariant under translations and scalings. Huisken's monotonicity formula [14] implies that $\lambda(F_t)$ is non-increasing if $\{F_t\}$ satisfies the MCF, and is constant if and only if $\{F_t\}$ is self-shrinking. Analogous to the entropy stability introduced in [6], we define Lagrangian entropy stability of a Lagrangian self-shrinker.

Definition 2.4. Let $F : \Sigma \rightarrow \mathbb{R}^{2n}$ be a self-shrinker. Then F is called Lagrangian entropy stable if $\lambda(\tilde{F}) \geq \lambda(F)$ for all Lagrangian immersions C^0 close to F .

In [6], it is proved that every \mathcal{F} -unstable embedded self-shrinking hypersurface which does not split off a line is entropy unstable. As observed in [1], the exact same proof works for any codimension. According to [20], the second variation formula for the \mathcal{F} -functional at a closed self-shrinker can be rewritten in terms of the closed 1-form dual to the Lagrangian variation field. Therefore, when $F : \Sigma \rightarrow \mathbb{R}^{2n}$ is a Lagrangian \mathcal{F} -unstable self-shrinker, there is a closed 1-form α on Σ so that $\mathcal{F}''(\alpha) < 0$ for all variations

(x_s, t_s) of $(0, 1)$. To proceed from the Lagrangian \mathcal{F} -instability to the Lagrangian entropy instability, one needs to use the actual family F_s of Lagrangian immersions coming from the Lagrangian variation. By Lemma 2.1, there is a Lagrangian variation $\{F_s\}$ that corresponds to α . By taking a family of diffeomorphism $\phi_s : \Sigma \rightarrow \Sigma$, we can further assume that $\{F_s\}$ is a family of normal variations. Thus the same proof of Theorem 0.15 in [6] can be carried over to show that $F : \Sigma \rightarrow \mathbb{R}^{2n}$ is also Lagrangian entropy unstable. We omit the proof here.

Theorem 2.2. *Let Σ be compact and $F : \Sigma \rightarrow \mathbb{R}^{2n}$ be an immersed Lagrangian self-shrinker. If F is Lagrangian \mathcal{F} -unstable, then it is also Lagrangian entropy unstable. In particular, there is a Lagrangian immersion $\hat{F} : \Sigma \rightarrow \mathbb{R}^{2n}$ so that $\lambda(F) > \lambda(\hat{F})$. Moreover, \hat{F} can be chosen to be arbitrarily close to F , in the sense of smallness of $\|F - \hat{F}\|_{C^k}$ for all k .*

3. LAGRANGIAN SELF-SHRINKING TORI WITH SMALL AREA

In this section, we prove Theorem 1.2. We will use a contradiction argument, and by doing so we need to extend the compactness theorems (Theorems 1.2 and 1.3 in [5]) to *branched* conformally immersed Lagrangian self-shrinking surfaces. This extension will be done for any area upper bound (not necessarily small), and beside Theorem 1.2, it will also be used in the next section in the proof of Theorem 1.1. Yet, for the construction of a piecewise Lagrangian MCF of Lagrangian torus, the compactness results in [5] suffice.

Then we will combine the compactness results with the factorization result of Lamm and Schätzle [17] concerning conformal immersion of torus into \mathbb{R}^4 with Willmore energy 8π to conclude the theorem.

Remark 1. In [5], a branched self-shrinker is defined as a branched immersion $F : \Sigma \rightarrow \mathbb{R}^N$ which satisfies

$$F^\perp = -\vec{H},$$

as opposed to (2.2), which has an extra $1/2$ factor. Note that both definitions are common in the literatures and differ only by a scaling of the branched immersion. As a result, it should be pointed out that the constants that appear in this section are slightly different from those in [5].

As in [5], we view a branched self-shrinker $F : \Sigma \rightarrow \mathbb{R}^4$ in \mathbb{R}^4 as a harmonic map from (Σ, h) to (\mathbb{R}^4, G) . Here h is the conformal structure on Σ such that F is conformal with respect to h , and G , where G is the metric on \mathbb{R}^4 given by

$$(3.1) \quad G_{ij} = e^{-\frac{|x|^2}{4}} \delta_{ij},$$

where δ_{ij} represents the standard Euclidean metric on \mathbb{R}^4 . Then we use general harmonic map theories from [23] and [4]. In order to use these results, as in [5], we need to show that the self-shrinkers with a uniform area upper bound stay in a bounded domain in (\mathbb{R}^4, G) . In particular, we need the following lemma, which extends Lemma 4.1 in [5] to allow branch points. The proof is almost the same as that of Lemma 4.1 in [5], except extra effort needs to be given at the branch points.

Lemma 3.1. *Let F be a compact branched conformal self-shrinker in \mathbb{R}^4 . Then the image of F lies in a ball of radius R_0 centered at the origin in \mathbb{R}^4 , where R_0 depends only on $\mu(F)$.*

Proof. Let $F : \Sigma \rightarrow \mathbb{R}^4$ be a branched conformally immersed self-shrinker. By (2.2), the equation

$$(3.2) \quad \Delta_g |F|^2 = -|F^\perp|^2 + 4$$

holds on $\Sigma \setminus B$, where $g = F^*\langle \cdot, \cdot \rangle$ and B is the finite branch locus.

First, we show that F must intersect the closed ball centered at the origin of \mathbb{R}^4 with radius 2. Since F is a branched conformal immersion, there is a nonnegative smooth function φ and a smooth metric g_0 on Σ compatible with the conformal structure h so that $g = \varphi g_0$. Therefore

$$\varphi \Delta_g = \Delta_{g_0}$$

and by (3.2),

$$(3.3) \quad \Delta_{g_0} |F|^2 = \varphi (-|F^\perp|^2 + 4).$$

Unlike (3.2), (3.3) is satisfied everywhere on Σ , as both sides of the equation are continuous and B is finite. Since Σ is compact, the smooth function $|F|^2$ attains its minimum, say at $x_0 \in \Sigma$. Since F is a minimal immersion in (\mathbb{R}^4, G) , the tangential component F^\top is well defined at a branch point and $F^\top(x_0) = 0$. If F is immersed at x_0 , by using (3.2) as in [5], the weak maximum principle shows that $|F(x_0)|^2 \leq 4$ and we are done. Thus we only need to rule out the case that F is branched at x_0 , $|F(x_0)|^2 > 4$ and there does not exist any immersed point $y \in \Sigma$ so that $|F(y)|^2 = |F(x_0)|^2$. Assume this case happens. Since the branch points are isolated, $|F|^2$ has a strict minimum at x_0 . Noting that $F^\top(x_0) = 0$ and $|F|^2 = |F^\top|^2 + |F^\perp|^2$, we have $|F^\perp(x)|^2 > 4$ in a neighbourhood of x_0 . By (3.3) we have $\Delta_{g_0} |F|^2 \leq 0$ in the neighbourhood. However, this contradicts the strong maximum principle, and we are done.

Next, we show that the extrinsic distance between any two points on the image of F is bounded above by a constant that depends only on the area upper bound. Note that

$$(3.4) \quad \Delta_g |F|^2 d\mu_g = d *_g d |F|^2.$$

and the Hodge star operator $*_g$ depends only on the conformal class of g , $\Delta_g |F|^2 d\mu_g$ is well-defined on Σ . Thus we integrate (3.2) and use (2.2) to get

$$(3.5) \quad \mathcal{W}(F) := \frac{1}{4} \int_{\Sigma} |\vec{H}|^2 d\mu = \frac{1}{4} \mu(F).$$

One also note that Simon's diameter estimate [27] holds for 2-varifolds with square integrable generalized mean curvature ((A.16) in [16]). Thus there is a constant C such that

$$\left(\frac{\mu(F)}{\mathcal{W}(F)} \right)^{\frac{1}{2}} \leq \text{diam} F(\Sigma) \leq C (\mu(F) \mathcal{W}(F))^{\frac{1}{2}},$$

where

$$\text{diam} F(\Sigma) := \sup_{x, y \in \Sigma} |F(x) - F(y)|.$$

Together with (3.5), we see that

$$\text{diam}F(\Sigma) \leq \frac{1}{2}C\mu(F).$$

It follows that the image of F lies in $B(R_0)$ for some R_0 depending only on the area upper bound. \square

Let $\{F_n : \Sigma \rightarrow \mathbb{R}^4\}$ be a sequence of compact Lagrangian branched conformal self-shrinkers with uniform upper bound Λ . Lemma 3.1 implies that the images of F_n lie in a bounded region B_R in \mathbb{R}^4 . The Riemannian space (B_R, G) can be isometrically embedded into a compact Riemannian manifold (N, g) , by Lemma 4.2 in [5]. We can assume that $\{F_n\}$ is also a sequence of harmonic mappings from (Σ, h_n) to the compact Riemannian manifold (N, g) . Using the same argument in [5], we extend Theorems 1.2 and 1.3 in [5] to the branched immersions.

We now state the main compactness result.

Theorem 3.1. *Let $F_n : \Sigma \rightarrow \mathbb{R}^4$ be a sequence of branched conformally immersed Lagrangian self-shrinkers with a uniform area upper bound Λ .*

- (1) *If $\Sigma = \mathbb{T}$ is topologically a torus, then by passing to subsequence if necessary, there is a conformal structure h on \mathbb{T} so that $F_n : (\mathbb{T}, h_n) \rightarrow \mathbb{R}^4$ converge smoothly to a branched conformally immersed Lagrangian self-shrinker $F : (\mathbb{T}, h) \rightarrow \mathbb{R}^4$ and h_n converge to h .*
- (2) *If Σ is a closed surface of genus $g \geq 2$ and assume additionally that the conformal structures h_n converge to a conformal structure h on Σ . Then by passing to subsequence if necessary, $F_n : (\Sigma, h_n) \rightarrow \mathbb{R}^4$ converges smoothly to a branched conformally immersed Lagrangian self-shrinker $F : (\Sigma, h) \rightarrow \mathbb{R}^4$.*

Note that in [5] we first show (2) and then argue that the conformal structures must converge when $\Sigma = \mathbb{T}$ because degeneration of conformal structures in the genus 1 case would create Lagrangian self-shrinking 2-spheres (possibly branched) but this would violate our rigidity result (see [5] for the details).

In particular, (1) in Theorem 3.1 implies (1) in Theorem 1.2. The proof of the remaining parts of Theorem 1.2 will be divided into the following results. We recall that \mathfrak{X}_Λ stands for the space of branched conformally immersed Lagrangian self-shrinking tori of area no larger than Λ .

Proposition 3.1. *There is a positive number ϵ_0 so that if $F \in \mathfrak{X}_{32\pi+\epsilon_0}$, then F is immersed.*

Proof. Arguing by contradiction, we assume that there is a sequence $F_n : \mathbb{T} \rightarrow \mathbb{R}^4$ of branched conformal Lagrangian self-shrinking tori so that

$$(3.6) \quad \liminf_{n \rightarrow \infty} \mu(F_n) \leq 32\pi$$

and each F_n has a nonempty set of branch points. Using Theorem 3.1, by passing to a subsequence if necessary, the sequence $\{F_n\}$ converges smoothly to a branched conformal Lagrangian self-shrinking torus $F_\infty : \mathbb{T} \rightarrow \mathbb{R}^4$. Let B_n be the set of branch points of F_n . Since \mathbb{T} is compact, again by passing to a subsequence if necessary, there is a sequence

$\{p_n\}$, where $p_n \in B_n$ for each $n \in \mathbb{N}$, so that $p_n \rightarrow p \in \mathbb{T}$. As $DF_n(p_n) = 0$ for all $n \in \mathbb{N}$ and the convergence $F_n \rightarrow F_\infty$ is smooth, $DF(p) = 0$ and so p is a branch point of F_∞ , where DF, DF_n are the differentials of F, F_n , respectively. By the theorem of Li and Yau (Theorem 6 in [19], see also the appendix in [16] for the generalization to branched immersions), since F_∞ is not embedded,

$$(3.7) \quad \mathcal{W}(F_\infty) \geq 8\pi.$$

On the other hand, from (3.6) and Theorem 1 in [4],

$$\mu(F_\infty) \leq \liminf \mu(F_n) \leq 32\pi.$$

Together with (3.7) and (3.5) we have $\mathcal{W}(F_\infty) = 8\pi$. Since \mathcal{F}_∞ has a branch point, Proposition 2.3 in [17] implies that F_∞ factors through a branched conformal immersion $g : \mathbb{T} \rightarrow \mathbb{S}^2$. It follows that there is a branch conformal Lagrangian self-shrinking sphere $h : \mathbb{S}^2 \rightarrow \mathbb{R}^4$ so that $F_\infty = h \circ g$. However, by Theorem 1 in [5], such a h does not exist. This contradicts the existence of the sequence $\{F_n\}$. The proposition is now proved. \square

Proposition 3.1 and Theorem 3.1 lead to

Theorem 3.2. *Let ϵ_0 be as in Proposition 3.1. Then the space of all Lagrangian **immersed** self-shrinking tori with area less than or equal to $32\pi + \epsilon_0$ is compact.*

Next we prove part (3) in Theorem 1.2.

Corollary 3.1. *(Curvature Estimates) There is $C_0 > 0$ so that if $F : \mathbb{T} \rightarrow \mathbb{R}^4$ is a Lagrangian immersed self-shrinking torus with area less than or equals to $32\pi + \epsilon_0$, then the second fundamental form of F is bounded by C_0 .*

Proof. Assume this were not true. Then there is a sequence $F_n : \mathbb{T} \rightarrow \mathbb{R}^4$ of Lagrangian immersed self-shrinking tori with area less than $32\pi + \epsilon_0$ so that

$$(3.8) \quad \max_{F_n(\mathbb{T})} |A_n| \rightarrow \infty,$$

where A_n is the second fundamental form of the immersion F_n . Using Theorem 3.2, a subsequence of $\{F_n\}$ converges smoothly to an immersed self-shrinker F_∞ . In particular, we have

$$(g_n)_{ij} = \frac{\partial F_n}{\partial x_i} \cdot \frac{\partial F_n}{\partial x_j} \longrightarrow \frac{\partial F_\infty}{\partial x_i} \cdot \frac{\partial F_\infty}{\partial x_j} = (g_\infty)_{ij}, \text{ as } n \rightarrow \infty.$$

Since g_∞ is positive definite as F_∞ is immersed, there is a positive number C so that $g_n \geq C\delta_{ij}$ for all n . So g_n^{-1} are uniformly bounded. Hence

$$\max_{F_n(\mathbb{T})} |A_n|^2 = \max_{F_n(\mathbb{T})} g_n^{ij} g_n^{kl} \langle (A_n)_{ik}, (A_n)_{jl} \rangle$$

are uniformly bounded and (3.8) is impossible. \square

To finish the proof of Theorem 1.2, it remains to prove the second part in (2).

Proposition 3.2. *There is a positive constant $\epsilon_1 \leq \epsilon_0$ so that if $F \in \mathfrak{X}_{32\pi+\epsilon_1}$, then F is embedded.*

Proof. As in the proof of Corollary 3.1, assume the contrary that there is a sequence $\{F_n\}$ of immersed, non-embedded Lagrangian self-shrinking tori with $\mu(F_n) \leq 32\pi + \epsilon_0$ and $\mu(F_n) \rightarrow 32\pi$. By Theorem 3.2, after passing to a subsequence if necessary, $\{F_n\}$ converges smoothly to an immersed Lagrangian self-shrinking torus $F_\infty : \mathbb{T} \rightarrow \mathbb{R}^4$ with area $\mu(F_\infty) = 32\pi$. By (3.5), the Willmore energy of F_∞ is 8π . Since each F_n is non-embedded, there are distinct points $p_n, q_n \in \mathbb{T}$ so that

$$(3.9) \quad F_n(p_n) = F_n(q_n).$$

As \mathbb{T} is compact, we may assume $p_n \rightarrow p$ and $q_n \rightarrow q$. Taking $n \rightarrow \infty$ in (3.9), we have $F_\infty(p) = F_\infty(q)$. First of all, we must have $p = q$: Indeed, if $p \neq q$, then F_∞ is not embedded and that contradicts Theorem 2.2 in [17], which states that any immersion $F : \mathbb{T} \rightarrow \mathbb{R}^4$ with $\mathcal{W}(F) = 8\pi$ has to be embedded.

Let d_n be the distance function on \mathbb{T} induced by the pullback metric $F_n^*\langle \cdot, \cdot \rangle$. As $p = q$ and $\{F_n\}$ converges smoothly to F , we have $\ell_n := d_n(p_n, q_n) \rightarrow 0$ as $n \rightarrow \infty$. Let $\eta_n : [0, \ell_n] \rightarrow \mathbb{T}^2$ be a shortest geodesics in $(\mathbb{T}, F_n^*\langle \cdot, \cdot \rangle)$ joining p_n to q_n . Since $F_n(\eta_n(0)) = F_n(\eta_n(\ell_n))$, $F_n \circ \eta_n : [0, \ell_n] \rightarrow \mathbb{R}^4$ is a closed curve in \mathbb{R}^4 with length ℓ_n . Let $\gamma_n : [0, \ell_n] \rightarrow \mathbb{R}^4$ be the translation $\gamma_n(t) = F_n \circ \eta_n(t) - F_n(p_n)$. Then each γ_n is parameterized by arc length and $\gamma_n(0) = \gamma_n(\ell_n) = 0 \in \mathbb{R}^4$. Using the following simple estimates

$$\begin{aligned} \ell_n &= \int_0^{\ell_n} \langle \gamma'_n(t), \gamma'_n(t) \rangle dt \\ &= - \int_0^{\ell_n} \langle \gamma_n(t), \gamma''_n(t) \rangle dt + \langle \gamma_n(\ell_n), \gamma'_n(\ell_n) \rangle - \langle \gamma_n(0), \gamma'_n(0) \rangle \\ &= - \int_0^{\ell_n} \langle \gamma_n(t), \gamma''_n(t) \rangle dt \\ &\leq \int_0^{\ell_n} |\gamma_n(t)| \cdot |\gamma''_n(t)| dt \\ &\leq \ell_n \int_0^{\ell_n} |\gamma''_n(t)| dt, \end{aligned}$$

we obtain

$$\int_0^{\ell_n} |\gamma''_n(t)| dt \geq 1.$$

Since $\ell_n \rightarrow 0$, the above inequality implies that there is $s_n \in [0, \ell_n]$ so that $|\gamma''_n(s_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Since η_n is a geodesic on $(\mathbb{T}, F_n^*\langle \cdot, \cdot \rangle)$,

$$\gamma''_n = (F_n \circ \eta_n)'' = \nabla_{\eta'_n}^n \eta'_n + A_n(\eta'_n, \eta'_n) = A_n(\eta'_n, \eta'_n),$$

where ∇^n is the Levi-Civita connection on $(\mathbb{T}, F_n^*\langle \cdot, \cdot \rangle)$ and A_n is the second fundamental form of $F_n(\mathbb{T})$ in \mathbb{R}^4 . Thus

$$|\gamma''_n(t)| \leq |A_n(\eta_n(t))|$$

and this implies

$$\max_{F_n(\mathbb{T})} |A_n|^2 \rightarrow \infty$$

as $n \rightarrow \infty$. However, this is impossible by Corollary 3.1. \square

4. A ŁOJASIEWICZ-SIMON TYPE GRADIENT INEQUALITY FOR BRANCHED SELF-SHRINKING TORI

In the last section we show that with a small area bound, all Lagrangian self-shrinking tori are immersed at least. This makes it much easier to study the space $\mathfrak{X}_{32\pi+\epsilon_0}$, as all nearby Lagrangian self-shrinking tori can be deformed to each other by using the normal vectors fields. However, it is difficult in general to relate two nearby branched conformal immersions, even if they are C^k -close when treated as mappings to the Euclidean space. In particular, it seems difficult to extend the perturbation procedure as in [6, 20], where the stability condition is described by using the normal vector fields, to *branched* conformal self-shrinkers, .

In this section, we show that the entropy λ is locally a constant function in the space of branched conformal compact self-shrinking tori $F : \mathbb{T} \rightarrow \mathbb{R}^4$. To do this we derive a Łojasiewicz-Simon type gradient inequality for branched conformal self-shrinking tori $F : \mathbb{T} \rightarrow \mathbb{R}^4$. In the genus one case, the explicit expression of the conformal structures in the Teichmüller space makes the computation and the real analyticity of the functional \mathcal{E} transparent. Once this is done, together with the compactness of \mathfrak{X}_Λ , we conclude the proof of Theorem 1.1.

4.1. A Fredholm operator of index zero. Let (Σ, g) be a compact Riemannian surface and (M, h) a Riemannian manifold. Given a C^1 mapping $F : \Sigma \rightarrow M$, the energy of F is given by

$$\mathcal{E}_{g,h}(F) = \frac{1}{2} \int_{\Sigma} e_{g,h}(F) d\mu_g,$$

where $e_{g,h}(F)$ is the norm of the differential $DF_x : T_x \Sigma \rightarrow T_{F(x)} M$. Locally it is given by

$$e_{g,h}(F) = g^{ij} h_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j}.$$

For a fixed h , define $\mathcal{E} : C^1(\Sigma, M) \times \{g : g \text{ is a Riemannian metric on } \Sigma\} \rightarrow \mathbb{R}$ by

$$\mathcal{E}(F, g) = \mathcal{E}_{g,h}(F).$$

Lemma 4.1. *If $F : (\Sigma, g) \rightarrow (M, h)$ is conformal, then g is a critical point of \mathcal{E} with respect to all its smooth variations g_s , where $g_0 = g$. That is,*

$$\left. \frac{d}{ds} \mathcal{E}_{g_s,h}(F) \right|_{s=0} = 0.$$

Proof. Let g_s be a family of smooth metrics on Σ so that $g_0 = g$ and $\dot{g} = \frac{d}{ds} g_s|_{s=0}$. Then

$$\begin{aligned} \left. \frac{d}{ds} \left(g^{ij} \sqrt{\det g} \right) \right|_{s=0} &= -g^{ik} g^{jl} \dot{g}_{kl} \sqrt{\det g} + \frac{1}{2} g^{ij} \sqrt{\det g} g^{kl} \dot{g}_{kl} \\ &= \left(\frac{1}{2} g^{kl} g^{ij} - g^{ik} g^{jl} \right) \dot{g}_{kl} \sqrt{\det g} \end{aligned}$$

Thus

$$(4.1) \quad \left. \frac{d}{ds} e_{g_s, h}(F) d\mu_g \right|_{s=0} = \left(\frac{1}{2} g^{kl} g^{ij} - g^{ik} g^{jl} \right) \dot{g}_{kl} h_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} d\mu_g$$

Since F is conformal,

$$(4.2) \quad h_{\alpha\beta} \frac{\partial F^\alpha}{\partial x^i} \frac{\partial F^\beta}{\partial x^j} = \varphi g_{ij}$$

for some function φ on Σ . Put (4.2) into (4.1) and use $g^{ij} g_{ij} = 2$ since Σ is two dimensional, we see that $\left. \frac{d}{ds} \mathcal{E}_{g_s, h}(F) \right|_{s=0} = 0$, as claimed. \square

On the other hand, recall that a branched minimal immersion is (weakly) conformal and harmonic, and we have the following ([30], Theorem 1.8)

Proposition 4.1. *If u is critical map of \mathcal{E} with respect to the variations of u and the conformal structures on Σ , then u is a branched minimal immersion.*

Let U be an open subset in the upper half space $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im } \tau > 0\}$. It is well-known that the upper half space represents the Teichmüller space of the standard torus $\mathbb{T} = \mathbb{R}^2 / \{1, i\}$ and we treat U as a local parameterization of the conformal structures on \mathbb{T} near a given one.

Let $0 < \alpha < 1$ be fixed. Define

$$\begin{aligned} \mathcal{U} &= C^{2, \alpha}(\mathbb{T}, \mathbb{R}^4) \times U, \\ \mathcal{C}^{k, \alpha} &= C^{k, \alpha}(\mathbb{T}, \mathbb{R}^4) \oplus \mathbb{R}^2 \\ \mathcal{W}^{k, p} &= W^{k, p}(\mathbb{T}, \mathbb{R}^4) \oplus \mathbb{R}^2 \\ \mathcal{L}^2 &= \mathcal{W}^{0, 2}. \end{aligned}$$

Note that $\mathcal{C}^{k, \alpha}, \mathcal{W}^{k, \alpha}$ are Banach spaces² with the norms

$$\begin{aligned} \|(\phi, \nu)\|_{k, \alpha} &= \|\phi\|_{C^{k, \alpha}} + |\nu|, \\ \|(\phi, \nu)\|_{\mathcal{W}^{k, p}} &= \|\phi\|_{W^{k, p}} + |\nu| \end{aligned}$$

respectively. When $(M, h) = (\mathbb{R}^4, G)$, where G is as in (3.1), the functional $\mathcal{E} : \mathcal{U} \rightarrow \mathbb{R}$ takes the form

$$(4.3) \quad \mathcal{E}(u, \tau) = \frac{1}{2} \int_{\mathbb{T}} e^{-\frac{|u|^2}{4}} |Du|_\tau^2 d\mu_\tau, \quad (u, \tau) \in \mathcal{U}.$$

Here g_τ is the metric on \mathbb{T} given by

$$(4.4) \quad g_\tau = \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix}^T \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix}$$

and

$$d\mu_\tau = d\mu_{g_\tau} = \sqrt{\det g_\tau} dx dy, \quad |Du|_\tau^2 = g_\tau^{ij} D_i u \cdot D_j u.$$

The metric g_τ is in the conformal class represented by τ , as it can be seen easily that g_τ is the pullback metric via the linear mapping from $\mathbb{T} = \mathbb{R}^2 / \{1, i\}$ to $\mathbb{R}^2 / \{1, \tau\}$. Note that

²All Banach spaces considered in this paper are real Banach spaces.

for each fixed τ , $\mathcal{E}(\cdot, \tau)$ is the Dirichlet energy functional of the mappings $u : (\mathbb{T}, g_\tau) \rightarrow (\mathbb{R}^4, G)$.

It is well-known [2] that minimal surface in (\mathbb{R}^4, G) corresponds to self-shrinking surfaces in \mathbb{R}^4 . Thus Lemma 4.1 and Proposition 4.1 imply the following

Proposition 4.2. *(u, τ) is a critical point of \mathcal{E} if and only if $u : (\mathbb{T}, g_\tau) \rightarrow \mathbb{R}^4$ is a branched conformal self-shrinking torus.*

Next we consider the L^2 -gradient $\mathcal{M} : \mathcal{U} \rightarrow \mathcal{C}^{0,\alpha}$ of \mathcal{E} . That is, we find for each $(u, \tau) \in \mathcal{U}$ an element $\mathcal{M}(u, \tau) \in \mathcal{C}^{0,\alpha}$ so that for all $(\phi, \nu) \in \mathcal{C}^{2,\alpha}$,

$$(4.5) \quad \left. \frac{d}{ds} \right|_{s=0} \mathcal{E}(u + s\phi, \tau + s\nu) = \langle \mathcal{M}(u, \tau), (\phi, \nu) \rangle_{u, \tau}.$$

Here we define

$$\langle \phi_1, \phi_2 \rangle_{u, \tau} = \int_{\mathbb{T}} \phi_1 \cdot \phi_2 e^{-\frac{|u|^2}{4}} d\mu_\tau$$

and

$$(4.6) \quad \langle (\phi_1, \nu_1), (\phi_2, \nu_2) \rangle_{u, \tau} = \langle \phi_1, \phi_2 \rangle_{u, \tau} + \nu_1 \cdot \nu_2.$$

Integrating by parts, we see that

$$(4.7) \quad \mathcal{M}(u, \tau) = \left(-g_\tau^{ij} e^{\frac{|u|^2}{4}} D_j (e^{-\frac{|u|^2}{4}} D_i u) - \frac{1}{4} |Du|_\tau^2 u, \nabla \mathcal{E}_\tau^u \right)$$

where $\mathcal{E}^u : U \rightarrow \mathbb{R}$ is given by $\mathcal{E}^u(\tau) = \mathcal{E}(u, \tau)$ and $\nabla \mathcal{E}_\tau^u$ is the gradient of \mathcal{E}^u at τ .

Let (u, τ) be a critical point of \mathcal{E} , that is, $\mathcal{M}(u, \tau) = 0$. Let

$$\mathcal{L} = \mathcal{L}_{(u, \tau)} : \mathcal{C}^{2,\alpha} \rightarrow \mathcal{C}^{0,\alpha}$$

be the Fréchet derivative of \mathcal{M} at (u, τ) . We will show that

$$(4.8) \quad \mathcal{L}(\phi, \nu) = (L\phi + \nabla_\nu B, (\nabla^2 \mathcal{E}_\tau^u)\nu + \langle \nabla B_\tau, \phi \rangle_{u, \tau})$$

where

$$(4.9) \quad L\phi = -g_\tau^{ij} e^{\frac{|u|^2}{4}} D_j (e^{-\frac{|u|^2}{4}} D_i \phi) - \frac{1}{4} |Du|_\tau^2 \phi + \frac{1}{2} g_\tau^{ij} D_j (u \cdot \phi) D_i u - \frac{1}{2} g_\tau^{ij} (D_j u \cdot D_i \phi) u$$

and $\nabla^2 \mathcal{E}_\tau^u$ is the Hessian of \mathcal{E}^u at τ ; furthermore, $B : U \rightarrow C^{0,\alpha}(\mathbb{T}, \mathbb{R}^4)$ is given by

$$(4.10) \quad B(\sigma) = -g_\sigma^{ij} \left(e^{\frac{|u|^2}{4}} D_j (e^{-\frac{|u|^2}{4}} D_i u) + \frac{1}{4} (D_i u \cdot D_j u) u \right)$$

and ∇B_τ denotes the Fréchet derivative of B at τ and $\nabla_\nu B_\tau$ stands for the Fréchet derivative of B at τ in the direction ν :

$$\nabla_\nu B_\tau = \left. \frac{d}{ds} \right|_{s=0} B(\tau + s\nu).$$

To derive (4.8), note that the two terms in the first component of (4.8) arise from direct differentiation of the first component of (4.7) with respect to ϕ and ν . To derive

the second component, note that $(\nabla^2 \mathcal{E}_\tau^u)\nu$ is just the directional derivative of $\nabla \mathcal{E}_\tau^u$ with respect to ν . Thus we need to show that $\nabla_\phi \nabla \mathcal{E}_\tau^u = \langle \nabla B_\tau, \phi \rangle_{u,\tau}$, where

$$\nabla_\phi \nabla \mathcal{E}_\tau^u = \frac{d}{ds} \Big|_{s=0} \nabla \mathcal{E}_\tau^{u+s\phi}.$$

Note

$$\nabla \mathcal{E}_\tau^u = \frac{1}{2} \int_{\mathbb{T}} (\nabla g_\tau^{ij}) e^{-\frac{|u|^2}{4}} (D_i u \cdot D_j u) d\mu_\tau + \frac{1}{4} \text{tr}(g_\tau^{-1} \nabla g_\tau) \mathcal{E}(u, \tau),$$

where the second term on the right comes from differentiating the volume form $d\mu_\tau$. Since (u, τ) is a critical point of \mathcal{E} , this term vanishes when we differentiate with respect to ϕ . Using this observation and integration by parts,

$$\begin{aligned} \nabla_\phi \nabla \mathcal{E}_\tau^u &= \frac{1}{2} \int_{\mathbb{T}} (\nabla g_\tau^{ij}) \nabla_\phi \left(e^{-\frac{|u|^2}{4}} (D_i u \cdot D_j u) \right) d\mu_\tau \\ &= \frac{1}{2} \int_{\mathbb{T}} (\nabla g_\tau^{ij}) \left(-\frac{1}{2} (u \cdot \phi) e^{-\frac{|u|^2}{4}} (D_i u \cdot D_j u) + 2e^{-\frac{|u|^2}{4}} (D_i u \cdot D_j \phi) \right) d\mu_\tau \\ &= - \int_{\mathbb{T}} (\nabla g_\tau^{ij}) \left(\frac{1}{4} (D_i u \cdot D_j u) u + e^{\frac{|u|^2}{4}} D_j (e^{-\frac{|u|^2}{4}} D_i) \cdot \phi \right) e^{-\frac{|u|^2}{4}} d\mu_\tau \\ &= \int_{\mathbb{T}} \nabla B_\tau \cdot \phi e^{-\frac{|u|^2}{4}} d\mu_\tau \\ &= \langle \nabla B_\tau, \phi \rangle_{u,\tau}. \end{aligned}$$

Thus (4.8) is shown.

Lemma 4.2. *Let (u, τ) be a critical point of \mathcal{E} . For all $(\phi, \nu), (\psi, \eta) \in \mathcal{C}^{2,\alpha}$, we have*

$$(4.11) \quad \langle \mathcal{L}(\phi, \nu), (\psi, \eta) \rangle_{u,\tau} = \langle (\phi, \nu), \mathcal{L}(\psi, \eta) \rangle_{u,\tau}.$$

Proof. Let $\phi, \psi \in C^{2,\alpha}(\mathbb{T}, \mathbb{R}^4)$, then from (4.8) and (4.10),

$$(4.12) \quad \begin{aligned} \langle L\phi, \psi \rangle_{u,\tau} &= \langle g_\tau^{ij} D_i \phi, D_j \psi \rangle_{u,\tau} - \frac{1}{4} \langle |Du|_\tau^2 \phi, \psi \rangle_{u,\tau} \\ &\quad + \frac{1}{2} \int g_\tau^{ij} D_j (u \cdot \phi) D_i u \cdot \psi e^{-\frac{|u|^2}{4}} d\mu_\tau - \frac{1}{2} \int g_\tau^{ij} (D_j u \cdot D_i \phi) (u \cdot \psi) e^{-\frac{|u|^2}{4}} d\mu_\tau. \end{aligned}$$

Integrating by parts for the third term on the right hand side in (4.12) gives

$$(4.13) \quad \begin{aligned} &\frac{1}{2} \int g_\tau^{ij} D_j (u \cdot \phi) D_i u \cdot \psi e^{-\frac{|u|^2}{4}} d\mu_\tau \\ &= -\frac{1}{2} \int g_\tau^{ij} (u \cdot \phi) (D_i u \cdot D_j \psi) e^{-\frac{|u|^2}{4}} d\mu_\tau - \frac{1}{2} \int (u \cdot \phi) g_\tau^{ij} D_j (e^{-\frac{|u|^2}{4}} D_i u) \cdot \psi d\mu_\tau. \end{aligned}$$

Since $\mathcal{M}(u, \tau) = 0$, we have by (4.7)

$$g_\tau^{ij} D_j (e^{-\frac{|u|^2}{4}} D_i u) = -\frac{1}{4} e^{-\frac{|u|^2}{4}} |Du|_\tau^2 u.$$

Putting this into (4.13), we have

$$\begin{aligned} \langle L\phi, \psi \rangle_{u,\tau} &= \langle g_\tau^{ij} D_i \phi, D_j \psi \rangle_{u,\tau} - \frac{1}{4} \langle |Du|_\tau^2 \phi, \psi \rangle_{u,\tau} \\ &\quad - \frac{1}{2} \int g_\tau^{ij} (u \cdot \phi) (D_i u \cdot D_j \psi) e^{-\frac{|u|^2}{4}} d\mu_{g_\tau} - \frac{1}{2} \int g_\tau^{ij} (D_j u \cdot D_i \phi) (u \cdot \psi) e^{-\frac{|u|^2}{4}} d\mu_\tau \\ &\quad + \frac{1}{8} \int (u \cdot \phi) (u \cdot \psi) |Du|_\tau^2 e^{-\frac{|u|^2}{4}} d\mu_\tau. \end{aligned}$$

Note that the right hand side is symmetric in ϕ and ψ . Thus

$$(4.14) \quad \langle L\phi, \psi \rangle_{u,\tau} = \langle \phi, L\psi \rangle_{u,\tau}, \quad \forall \phi, \psi \in C^{2,\alpha}(\mathbb{T}, \mathbb{R}^4).$$

Using this, we have

$$\begin{aligned} \langle \mathcal{L}(\phi, \nu), (\psi, \eta) \rangle_{u,\tau} &= \langle L\phi + \nabla_\nu B_\tau, \psi \rangle_{u,\tau} + (\nabla^2 \mathcal{E}_\tau^u \nu + \langle \nabla B_\tau, \psi \rangle_{u,\tau}) \cdot \eta \\ &= \langle L\phi, \psi \rangle_{u,\tau} + \langle \nabla_\nu B_\tau, \psi \rangle_{u,\tau} + \langle \nabla_\eta B_\tau, \phi \rangle_{u,\tau} + (\nabla^2 \mathcal{E}_\tau^u \nu) \cdot \eta \end{aligned}$$

Again, the right hand side is symmetric in (ϕ, ν) and (ψ, η) . We can now conclude the proof of the lemma. \square

Remark 2. Note that the apparent self-adjointness expression for \mathcal{L} in (4.11) only holds in $\mathcal{C}^{2,\alpha}$, and \mathcal{L} is an operator from $\mathcal{C}^{2,\alpha}$ to $\mathcal{C}^{0,\alpha}$. Nevertheless, (4.11) is useful in proving the following theorem.

Theorem 4.1. \mathcal{L} is a Fredholm operator of index zero at a critical point (u, τ) of \mathcal{E} .

Proof. The proof will be divided into several steps.

Step 1. We show that $\dim \ker \mathcal{L}$ is finite.

Consider the first component of \mathcal{L} ,

$$(4.15) \quad L\phi + \nabla_\nu B_\tau = 0.$$

This equation is bilinear in ϕ, ν . Let S be the subspace of \mathbb{R}^2 so that $\nu \in S$ if and only if (4.15) has a solution. If $S = \{(0, 0)\}$, then $\dim \ker \mathcal{L} = \dim \ker L < \infty$ since L is elliptic. If not, let $\{\nu_i\}$ be a basis of S . Pick $\phi_i \in C^{2,\alpha}(\mathbb{T}, \mathbb{R}^4)$ so that ϕ_i satisfies (4.15) with $\nu = \nu_i$. Let $(\phi, \nu) \in \ker \mathcal{L}$. Then $\nu \in S$. Write $\nu = \sum_i s^i \nu_i$ for some $s^i \in \mathbb{R}$. Then $\phi - s^i \phi_i \in \ker L$ and thus

$$(\phi, \nu) = (\phi_0, 0) + \sum_i s^i (\phi_i, \nu_i)$$

for some $\phi_0 \in \ker L$. Again, due to the ellipticity of L , $\dim \ker L$ is finite, hence $\ker \mathcal{L}$ is finite dimensional.

Step 2. \mathcal{L} has finite dimensional cokernel. Moreover, $\dim \ker \mathcal{L} = \dim \operatorname{coker} \mathcal{L}$.

We will show that the mapping

$$(4.16) \quad \ker \mathcal{L} \hookrightarrow \mathcal{C}^{2,\alpha} \hookrightarrow \mathcal{C}^{0,\alpha} \xrightarrow{\pi} \operatorname{coker} \mathcal{L}$$

is bijective, where π is the projection to the quotient $\operatorname{coker} \mathcal{L} = \mathcal{C}^{0,\alpha} / \operatorname{Im} \mathcal{L}$.

Firstly, if $(\psi_1, \eta_1), (\psi_2, \eta_2) \in \ker \mathcal{L}$ represent the same element in $\operatorname{coker} \mathcal{L}$, then there is $(\phi, \nu) \in \mathcal{C}^{2,\alpha}$ so that

$$(\psi, \eta) := (\psi_1 - \psi_2, \eta_1 - \eta_2) = \mathcal{L}(\phi, \nu).$$

Using (4.11),

$$\langle (\psi, \eta), (\psi, \eta) \rangle_{u, \tau} = \langle \mathcal{L}(\phi, \nu), (\psi, \eta) \rangle_{u, \tau} = \langle (\phi, \nu), \mathcal{L}(\psi, \eta) \rangle_{u, \tau} = 0.$$

Thus $(\psi, \eta) = 0$ and so the mapping $\ker \mathcal{L} \rightarrow \operatorname{coker} \mathcal{L}$ defined in (4.16) is injective.

Secondly, we show that the mapping $\ker \mathcal{L} \rightarrow \operatorname{coker} \mathcal{L}$ is surjective. Let $\overline{\operatorname{Im} \mathcal{L}}$ be the L^2 closure of the image of \mathcal{L} in \mathcal{L}^2 with respect to the inner product defined in (4.6). Let $(\psi, \eta) \in \mathcal{C}^{0, \alpha}$ represents an element in $\operatorname{coker} \mathcal{L}$. We decompose (ψ, η) into the component in $\overline{\operatorname{Im} \mathcal{L}}$ and $\overline{\operatorname{Im} \mathcal{L}}^\perp$. That is,

$$(4.17) \quad (\psi, \eta) = (\psi^\top, \eta^\top) + (\psi^\perp, \eta^\perp)$$

for some $\psi^\top, \psi^\perp \in L^2(\mathbb{T}, \mathbb{R}^4)$. Note that

$$\langle (\psi^\perp, \eta^\perp), \mathcal{L}(\phi, \nu) \rangle_{u, \tau} = 0$$

for all $(\phi, \nu) \in \mathcal{C}^{2, \alpha}$. Letting $\nu = 0$ and using (4.8), we have

$$\langle L\phi, \psi^\perp \rangle_{u, \tau} + \langle \nabla_{\eta^\perp} B_\tau, \phi \rangle_{u, \tau} = 0, \quad \forall \phi \in C^{2, \alpha}(\mathbb{T}, \mathbb{R}^4).$$

Note that the above equation is of the form

$$\int (-g_\tau^{ij} D_{ij} \phi + \mathcal{A}^i D_i \phi + \mathcal{B} \phi) \cdot \psi^\perp dx dy = \int \mathcal{F} \cdot \phi dx dy,$$

where $\mathcal{A}^i = (\mathcal{A}_{\beta\gamma}^i)$ and $\mathcal{B} = (\mathcal{B}_{\beta\gamma})$ are (4×4) -matrix-valued smooth functions and $\mathcal{F} = (\mathcal{F}_\beta)$ is a \mathbb{R}^4 -valued smooth function. If we choose $\phi = (\rho, 0, 0, 0)$, where $\rho \in C^\infty(\mathbb{T}, \mathbb{R})$, we have

$$(4.18) \quad \int (-g_\tau^{ij} D_{ij} \rho + \mathcal{A}_{11}^i D_i \rho + \mathcal{B}_{11} \rho) \psi_1^\perp dx dy = D(\rho),$$

where

$$(4.19) \quad D(\rho) = - \int \sum_{k \neq 1} \mathcal{A}_{k1}^i \psi_k^\perp D_i \rho dx dy - \int \sum_{k \neq 1} \mathcal{B}_{k1} \psi_k^\perp \rho dx dy + \int \mathcal{F}_1 \rho dx dy.$$

Since ψ_k^\perp are in L^2 (noting that the L^2 spaces with respect the area elements $e^{-\frac{|u|^2}{4}} d\mu_\tau$ and $dx dy$ coincide over \mathbb{T}), as a distribution, D is in H_{loc}^{-1} . Thus the Elliptic Regularity Theorem (Theorem 6.33 in [11]) asserts $\psi_1^\perp \in H_{loc}^1$. Similarly, we have $\psi_k^\perp \in H_{loc}^1$ for $k = 2, 3, 4$. Putting this information into (4.19), we see that $D \in H_{loc}^0$, and in turn, this implies $\phi_1^\perp \in H_{loc}^2$ by the Elliptic Regularity Theorem again. By a standard bootstrapping argument and the Sobolev embedding theorem, we see that $\psi^\perp \in C^{2, \alpha}$ (in fact, smooth). Using (4.11) and the definition of (ψ^\perp, η^\perp) , we have

$$(\mathcal{L}(\psi^\perp, \eta^\perp), (\phi, \nu))_{u, \tau} = 0$$

for all $(\phi, \nu) \in \mathcal{C}^{2, \alpha}$, thus

$$\mathcal{L}(\psi^\perp, \eta^\perp) = 0.$$

The smoothness of (ψ^\perp, η^\perp) asserts $(\psi^\top, \eta^\top) \in \mathcal{C}^{0, \alpha}$. If we can show that

$$(4.20) \quad (\psi^\top, \eta^\top) \in \operatorname{Im} \mathcal{L},$$

then $\pi(\psi, \eta) = \pi(\psi^\perp, \eta^\perp)$ by (4.17) and it follows that the mapping $\ker \mathcal{L} \rightarrow \operatorname{coker} \mathcal{L}$ defined in (4.16) is surjective and we are done. To show (4.20), recall that $(\psi^\top, \eta^\top) \in$

$\overline{\text{Im } \mathcal{L}}$. Thus there is a sequence $(\phi_n, \nu_n) \in \mathcal{C}^{2,\alpha}$ so that $\mathcal{L}(\phi_n, \nu_n) \rightarrow (\psi^\top, \eta^\top)$ in \mathcal{L}^2 . Using the \mathcal{L}^2 inner product, we decompose (ϕ_n, ν_n) into

$$(4.21) \quad (\phi_n, \nu_n) = (\phi_n^K, \nu_n^K) + (\phi_n^P, \nu_n^P),$$

where $(\phi_n^K, \nu_n^K) \in \ker \mathcal{L}$ and $(\phi_n^P, \nu_n^P) \in \ker \mathcal{L}^\perp$. Then by setting

$$(4.22) \quad (\psi_n, \eta_n) = \mathcal{L}(\phi_n^P, \nu_n^P)$$

and using $\mathcal{L}(\phi_n^K, \nu_n^K) = 0$, we have

$$\begin{aligned} (\psi_n, \eta_n) &= \mathcal{L}(\phi_n^P, \nu_n^P) \\ &= \mathcal{L}(\phi_n, \nu_n) \\ &\xrightarrow{\mathcal{L}^2} (\psi^\top, \eta^\top). \end{aligned}$$

The convergence above in particular implies that $\|\psi_n\|_{L^2} \leq C$ for some constant C . From the first component of (4.8), which is

$$L\phi_n^P = \psi_n - \nabla_{\nu_n^P} B_\tau,$$

the standard elliptic estimates (Theorem 9.11 in [12]) implies that there are constants $C', C'', C''' > 0$ so that

$$\begin{aligned} (4.23) \quad \|\phi_n^P\|_{W^{2,2}} &\leq C' (\|\phi_n^P\|_{L^2} + \|\psi_n - \nabla_{\nu_n^P} B_\tau\|_{L^2}) \\ &\leq C' (\|\phi_n^P\|_{L^2} + \|\psi_n\|_{L^2} + C'' |\nu_n^P|) \\ &\leq C''' (\|(\phi_n^P, \nu_n^P)\|_{\mathcal{L}^2} + 1). \end{aligned}$$

Next, we show that the sequence $\{\|(\phi_n^P, \nu_n^P)\|_{\mathcal{L}^2}\}$ is bounded. Assume not, then by taking a subsequence if necessary, we have $\|(\phi_n^P, \nu_n^P)\|_{\mathcal{L}^2} \rightarrow \infty$. Let

$$(4.24) \quad (\tilde{\phi}_n, \tilde{\nu}_n) = \frac{(\phi_n^P, \nu_n^P)}{\|(\phi_n^P, \nu_n^P)\|_{\mathcal{L}^2}}.$$

Then, as (ψ_n, η_n) converges to (ψ^\top, η^\top) in \mathcal{L}^2 ,

$$(4.25) \quad \mathcal{L}(\tilde{\phi}_n, \tilde{\nu}_n) = \frac{(\psi_n, \eta_n)}{\|(\phi_n^P, \nu_n^P)\|_{\mathcal{L}^2}} \xrightarrow{\mathcal{L}^2} 0.$$

Since $\|(\tilde{\phi}_n, \tilde{\nu}_n)\|_{\mathcal{L}^2} = 1$, we may assume $\tilde{\nu}_n \rightarrow \tilde{\nu}$ for some $\tilde{\nu} \in \mathbb{R}^2$. From (4.23) and (4.24), the sequence $\{\|\tilde{\phi}_n\|_{W^{2,2}}\}$ is bounded. Hence, again by taking subsequence if necessary, there is $\tilde{\phi} \in W^{2,2}(\mathbb{T}, \mathbb{R}^4)$ so that $\tilde{\phi}_n \rightarrow \tilde{\phi}$ in $W^{1,2}(\mathbb{T}, \mathbb{R}^4)$. Using (4.25), we have

$$\begin{cases} L\tilde{\phi} + \nabla_{\tilde{\nu}} B_\tau = 0 & \text{weakly in } W^{1,2}(\mathbb{T}, \mathbb{R}^4), \\ \nabla^2 \mathcal{E}_\tau^u \tilde{\nu} + \langle \nabla B_\tau, \tilde{\phi} \rangle_{u,\tau} = 0 \end{cases}$$

Since $\tilde{\phi} \in W^{2,2}(\mathbb{T}, \mathbb{R}^4)$, the first equation is actually satisfied strongly in $W^{2,2}(\mathbb{T}, \mathbb{R}^4)$. Since $\nabla_{\tilde{\nu}} B_\tau$ is smooth, by the elliptic regularity, $\tilde{\phi}$ is smooth. Thus $(\tilde{\phi}, \tilde{\nu}) \in \mathcal{C}^{2,\alpha}$ and $\mathcal{L}(\tilde{\phi}, \tilde{\nu}) = 0$, in other words, $(\tilde{\phi}, \tilde{\nu}) \in \ker \mathcal{L}$. On the other hand, since $(\tilde{\phi}_n, \tilde{\nu}_n) \rightarrow (\tilde{\phi}, \tilde{\nu})$ in \mathcal{L}^2 and $(\tilde{\phi}_n, \tilde{\nu}_n) \in \ker \mathcal{L}^\perp$, we also have $(\tilde{\phi}, \tilde{\nu}) \in \ker \mathcal{L}^\perp$. Thus $(\tilde{\phi}, \tilde{\nu}) = (0, 0)$. But this is

impossible as $\|(\tilde{\phi}, \tilde{\nu})\|_{\mathcal{L}^2} = 1$ since $(\tilde{\phi}_n, \tilde{\nu}_n) \rightarrow (\tilde{\phi}, \tilde{\nu})$ in \mathcal{L}^2 and $\|(\tilde{\phi}_n, \tilde{\nu}_n)\|_{\mathcal{L}^2} = 1$. The contradiction leads to the conclusion that the sequence $\{\|(\phi_n^P, \nu_n^P)\|_{\mathcal{L}^2}\}$ is bounded.

From (4.23), the sequence $\{\|(\phi_n^P, \nu_n^P)\|_{\mathcal{W}^{2,2}}\}$ is also bounded. By taking a subsequence if necessary, there is $(\phi, \nu) \in \mathcal{W}^{2,2}$ so that $(\phi_n^P, \nu_n^P) \rightarrow (\phi, \nu)$ in $\mathcal{W}^{1,2}$ and

$$\mathcal{L}(\phi, \nu) = (\psi^\top, \nu^\top).$$

The first component of this is given by

$$L\phi + \nabla_\nu B_\tau = \psi^\top.$$

Since $\phi \in W^{2,2}(\mathbb{T}, \mathbb{R}^4)$ and $\psi^\top \in C^{0,\alpha}(\mathbb{T}, \mathbb{R}^4)$, the standard elliptic regularity (Theorem 9.19 in [12]) implies that $\phi \in C^{2,\alpha}(\mathbb{T}, \mathbb{R}^4)$. Thus $(\phi, \nu) \in \mathcal{C}^{2,\alpha}$. This shows $(\psi^\top, \eta^\top) \in \text{Im } \mathcal{L}$. Therefore, the mapping $\ker \mathcal{L} \rightarrow \text{coker } \mathcal{L}$ is surjective.

Step 3. From the previous two steps, the bounded operator \mathcal{L} has finite dimensional kernel and cokernel so it is a Fredholm operator of

$$\text{index } \mathcal{L} = \dim \ker \mathcal{L} - \dim \text{coker } \mathcal{L} = 0.$$

This completes the proof of the theorem. \square

4.2. A Łojasiewicz-Simon type inequality. Next we prove a Łojasiewicz-Simon gradient inequality for compact branched self-shrinkers $F : \mathbb{T} \rightarrow \mathbb{R}^4$. As in [26], we use the Liapunov-Schmidt reduction argument and the classical Łojasiewicz inequality in [21]. See [10] for a Łojasiewicz-Simon inequality in the abstract setting and the related work in the reference therein.

Let

$$\Pi : \mathcal{L}^2 \rightarrow \ker \mathcal{L}$$

be the L^2 -projection with respect to the L^2 inner product:

$$(4.26) \quad \langle (\psi_1, \nu_1), (\psi_2, \nu_2) \rangle_{\mathcal{L}^2} = \int_{\mathbb{T}} \psi_1 \cdot \psi_2 \, dx dy + \nu_1 \cdot \nu_2$$

for all $(\psi_1, \nu_1), (\psi_2, \nu_2) \in \mathcal{L}^2$. Recall that $\ker \mathcal{L}$ is a finite dimensional subspace and $\ker \mathcal{L} \subset \mathcal{C}^\infty$. For all $k = 0, 1, 2, \dots$, we let

$$\Pi_k : \mathcal{C}^{k,\alpha} \rightarrow \mathcal{C}^{0,\alpha}$$

be the restriction of Π to $\mathcal{C}^{k,\alpha}$ composed with the inclusion $\ker \mathcal{L} \hookrightarrow \mathcal{C}^{0,\alpha}$.

Lemma 4.3. $\Pi_k : \mathcal{C}^{k,\alpha} \rightarrow \mathcal{C}^{0,\alpha}$ is a bounded linear operator for all nonnegative integers k . In particular, there is a positive constant C_α so that

$$(4.27) \quad \|\Pi_k(\psi, \nu)\|_{0,\alpha} \leq C_\alpha \|(\psi, \nu)\|_{k,\alpha}$$

for all $(\psi, \nu) \in \mathcal{C}^{k,\alpha}$.

Proof. Let $(\chi_1, \nu_1), \dots, (\chi_n, \nu_n) \in \ker \mathcal{L}$ be an orthonormal basis of the finite dimensional space $\ker \mathcal{L}$ with respect to the inner product in (4.26). Then for any $(\psi, \nu) \in \mathcal{L}^2$, we have

$$\Pi(\psi, \nu) = \sum_{i=1}^n \langle (\chi_i, \nu_i), (\psi, \nu) \rangle_{\mathcal{L}^2} (\chi_i, \nu_i).$$

Then we have

$$\begin{aligned} \|\Pi_k(\psi, \nu)\|_{0,\alpha} &\leq \sum_{i=1}^n |\langle (\chi_i, \nu_i), (\psi, \nu) \rangle_{\mathcal{L}^2}| \|(\chi_i, \nu_i)\|_{0,\alpha} \\ &\leq \left(\sum_{i=1}^n \|(\chi_i, \nu_i)\|_{0,\alpha} \right) \|(\psi, \nu)\|_{\mathcal{L}^2}. \end{aligned}$$

Note that we used the Cauchy-Schwarz inequality and that $\|(\chi_i, \nu_i)\|_{\mathcal{L}^2} = 1$. Since

$$\int_{\mathbb{T}} dx dy = 1,$$

we have

$$(4.28) \quad \|(\psi, \nu)\|_{\mathcal{L}^2} \leq \max_{\mathbb{T}^2} |\psi| + |\nu| \leq \|(\psi, \nu)\|_{k,\alpha}$$

for all nonnegative k . Now (4.27) follows with $C_\alpha = \sum_{i=1}^n \|(\chi_i, \nu_i)\|_{0,\alpha}$. \square

To simplify notations, in the sequel we use x, y and a, b to denote elements in $\mathcal{C}^{2,\alpha}$ and $\mathcal{C}^{0,\alpha}$ respectively. Let $x_c = (u, \tau)$ be a critical point of \mathcal{E} as before, that is $\mathcal{M}(x_c) = 0$.

Consider the mapping $\mathcal{N} : \mathcal{U} \rightarrow \mathcal{C}^{0,\alpha}$ given by

$$(4.29) \quad \mathcal{N}(x) = \mathcal{M}(x) + \Pi_2(x - x_c).$$

Since Π_2 is linear, the differential $D\mathcal{N}$ at x_c is given by

$$(4.30) \quad D\mathcal{N}_{x_c} = \mathcal{L} + \Pi_2.$$

Lemma 4.4. *$D\mathcal{N}_{x_c}$ is bijective and its inverse is bounded.*

Proof. First we show that $D\mathcal{N}_{x_c}$ is injective. Let $D\mathcal{N}_{x_c}(x) = 0$. Then by (4.30) we have

$$\mathcal{L}(x) = -\Pi_2 x.$$

Using (4.11), for all $y \in \ker \mathcal{L}$ we have

$$\langle \Pi_2 x, y \rangle_{u,\tau} = -\langle \mathcal{L}x, y \rangle_{u,\tau} = -\langle x, \mathcal{L}y \rangle_{u,\tau} = 0.$$

This means that $\Pi_2 x \in \ker \mathcal{L}$ is orthogonal to $\ker \mathcal{L}$. Therefore, $\Pi_2 x = 0$. Thus $\mathcal{L}x = 0$ and so $x \in \ker \mathcal{L}$. Hence $x = \Pi_2 x = 0$ and $D\mathcal{N}_{x_c}$ is injective.

By Theorem 4.1, \mathcal{L} is a Fredholm operator of index zero. Since Π_2 is bounded with a finite dimensional range, Π_2 is a compact operator and $D\mathcal{N}_{x_c} : \mathcal{C}^{2,\alpha} \rightarrow \mathcal{C}^{0,\alpha}$ is Fredholm with index zero (Theorem 5.10 in [24]). Together with the fact that $D\mathcal{N}_{x_c}$ is injective, $D\mathcal{N}_{x_c}$ is also surjective. Finally, the bounded inverse theorem (Theorem 3.8 in [24]) asserts that $D\mathcal{N}_{x_c}$ has a bounded inverse. \square

By the inverse function theorem for Banach spaces (Theorem 15.2 in [9]), since \mathcal{N} is C^1 (\mathcal{N} is even analytic: see the appendix), there are open neighbourhoods \mathcal{U}_1 of x_c in \mathcal{U} and \mathcal{V}_1 of 0 in $\mathcal{C}^{0,\alpha}$ so that $\mathcal{N} : \mathcal{U}_1 \rightarrow \mathcal{V}_1$ is invertible with a C^1 inverse Ψ . By shrinking $\mathcal{U}_1, \mathcal{V}_1$ if necessary, we assume that \mathcal{V}_1 is convex, \mathcal{U}_1 is contained in a convex

set $\mathcal{U}_2 \subset \mathcal{U}$ and (since \mathcal{M} and Ψ are C^1) there exist two positive constants M_1, M_2 so that

$$(4.31) \quad \begin{aligned} \|D\Psi(a)\|_{op} &\leq M_1, \quad \forall a \in \mathcal{V}_1, \\ \|D\mathcal{M}(x)\|_{op} &\leq M_2, \quad \forall x \in \mathcal{U}_2, \end{aligned}$$

where $\|\cdot\|_{op}$ denotes the operator norm for the corresponding operator. Using the Fundamental Theorem of Calculus, the above imply

$$(4.32) \quad \|\Psi(a) - \Psi(b)\|_{2,\alpha} \leq M_1 \|a - b\|_{0,\alpha}$$

for all $a, b \in \mathcal{V}_1$ and

$$(4.33) \quad \|\mathcal{M}(x) - \mathcal{M}(y)\|_{0,\alpha} \leq M_2 \|x - y\|_{2,\alpha}$$

for all $x, y \in \mathcal{U}_1$.

A main technical result in this section is the following Łojasiewicz-Simon type gradient inequality:

Theorem 4.2. *There is an open neighbourhood $\mathcal{W}_0 \subset \mathcal{U}$ of x_c , a positive constant C_2 and a constant $\theta \in (0, 1/2)$ depending on \mathcal{E} and x_c so that*

$$(4.34) \quad |\mathcal{E}(x) - \mathcal{E}(x_c)|^{1-\theta} \leq C_2 \|\mathcal{M}(x)\|_{0,\alpha}, \quad \forall x \in \mathcal{W}_0.$$

Proof. Since Π_0 is bounded, there is an open neighbourhood \mathcal{V}_0 of 0 so that $\mathcal{V}_0, \Pi_0 \mathcal{V}_0 \subseteq \mathcal{V}_1$. For all $a \in \mathcal{V}_0$, $\Pi_0 a \in \mathcal{V}_1$. Since \mathcal{U}_2 is convex, the line segment joining $\Psi(a)$ and $\Psi(\Pi_0 a)$ is in \mathcal{U}_2 . The Fundamental Theorem of Calculus and (4.5) yield

$$\begin{aligned} \mathcal{E}(\Psi(a)) - \mathcal{E}(\Psi(\Pi_0 a)) &= - \int_0^1 \frac{d}{dt} (\mathcal{E}(\Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a)))) dt \\ &= - \int_0^1 \langle \mathcal{M}(\Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a))), \Psi(\Pi_0 a) - \Psi(a) \rangle_{u_t, \tau_t} dt, \end{aligned}$$

where we write

$$(u_t, \tau_t) = \Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a)).$$

Using the Cauchy-Schwarz inequality, (4.28), (4.33) and $|t| \leq 1$,

$$(4.35) \quad \begin{aligned} &|\mathcal{E}(\Psi(a)) - \mathcal{E}(\Psi(\Pi_0 a))| \\ &\leq \|\mathcal{M}(\Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a)))\|_{\mathcal{L}^2} \|\Psi(\Pi_0 a) - \Psi(a)\|_{\mathcal{L}^2} \\ &\leq \|\mathcal{M}(\Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a)))\|_{0,\alpha} \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \\ &\leq (\|\mathcal{M}(\Psi(a))\|_{0,\alpha} + M_2 t \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha}) \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \\ &\leq (\|\mathcal{M}(\Psi(a))\|_{0,\alpha} + M_2 \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha}) \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \end{aligned}$$

On the other hand, since $a, \Pi_0 a \in \mathcal{V}_1$, by (4.32) we have

$$(4.36) \quad \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \leq M_1 \|\Pi_0 a - a\|_{0,\alpha}.$$

Using the definition of \mathcal{N} , Ψ and $\Pi_0 \Pi_2 = \Pi_2$,

$$(4.37) \quad a = \mathcal{N}(\Psi(a)) = \mathcal{M}(\Psi(a)) + \Pi_2(\Psi(a) - x_c)$$

$$\begin{aligned}
(4.38) \quad \Pi_0 a - a &= \Pi_0 a - \mathcal{M}(\Psi(a)) - \Pi_2(\Psi(a) - x_c) \\
&= \Pi_0(a - \Pi_2(\Psi(a) - x_c)) - \mathcal{M}(\Psi(a)).
\end{aligned}$$

Since Π_0 is bounded by Lemma 4.3,

$$\begin{aligned}
\|\Pi_0(a - \Pi_2(\Psi(a) - x_c))\|_{0,\alpha} &\leq C_\alpha \|a - \Pi_2(\Psi(a) - x_c)\|_{0,\alpha} \\
&= C_\alpha \|\mathcal{M}(\Psi(a))\|_{0,\alpha},
\end{aligned}$$

where in the last line we use (4.37) again. Combining this with (4.36) and (4.38), we are led to

$$(4.39) \quad \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \leq C_1 \|\mathcal{M}(\Psi(a))\|_{0,\alpha}$$

for all $a \in \mathcal{V}_0$ with $C_1 = M_1(C_\alpha + 1)$. Putting this into (4.35), we have

$$(4.40) \quad |\mathcal{E}(\Psi(a)) - \mathcal{E}(\Psi(\Pi_0 a))| \leq C_3 \|\mathcal{M}(\Psi(a))\|_{0,\alpha}^2$$

for all $a \in \mathcal{V}_0$ and for some $C_3 > 0$.

Let $f : \mathcal{V}_1 \cap \ker \mathcal{L} \rightarrow \mathbb{R}$ be defined by

$$(4.41) \quad f(a) = \mathcal{E}(\Psi(a)).$$

It is easy to show that \mathcal{E} , \mathcal{M} are analytic (a proof is given in the appendix for completeness). Since Π_2 is linear,

$$\mathcal{N} = \mathcal{M} + \Pi_2 - \Pi_2(x_c)$$

is analytic as well. Hence Ψ is analytic by the analytic version of inverse function theorem (Theorem 15.3 in [9]). Consequently, as a composition of analytic functions, f is also analytic, and it is defined on an open set in $\ker \mathcal{L}$, which is finite dimensional. The classical Łojasiewicz inequality [21] then implies that there is an open neighbourhood $\mathcal{V}_2 \subset \mathcal{V}_0$, constants $c > 0$ and $\theta \in (0, 1/2)$ so that

$$(4.42) \quad |f(\xi) - f(0)|^{1-\theta} \leq c |f'(\xi)|, \quad \forall \xi \in \mathcal{V}_2 \cap \ker \mathcal{L}.$$

Using (4.41) and (4.5), for all $b \in \mathcal{V}_1 \cap K$ we have

$$f'(b)(\cdot) = \langle \mathcal{M}(\Psi(b)), D\Psi_b(\cdot) \rangle_{u,\tau}.$$

Using (4.28), (4.33) and (4.39),

$$\begin{aligned}
(4.43) \quad |f'(\Pi_0 a)| &\leq M_1 \|\mathcal{M}(\Psi(\Pi_0 a))\|_{\mathcal{L}^2} \\
&\leq M_1 \|\mathcal{M}(\Psi(\Pi_0 a))\|_{0,\alpha} \\
&\leq M_1 (\|\mathcal{M}(\Psi(\Pi_0 a)) - \mathcal{M}(\Psi(a))\|_{0,\alpha} + \|\mathcal{M}(\Psi(a))\|_{0,\alpha}) \\
&\leq M_1 (M_2 \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} + \|\mathcal{M}(\Psi(a))\|_{0,\alpha}) \\
&\leq C_4 \|\mathcal{M}(\Psi(a))\|_{0,\alpha}
\end{aligned}$$

for some $C_4 > 0$. Now let $\mathcal{W}_0 = \Psi(\mathcal{V}_2)$. Thus for every $x \in \mathcal{W}_0$, there exists an $a \in \mathcal{V}_2$ such that $x = \Psi(a)$. By (4.43), the classical Łojasiewicz inequality (4.42) and (4.40),

$$\begin{aligned}
 (4.44) \quad C_4 c \|\mathcal{M}(x)\|_{0,\alpha} &\geq c |f'(\Pi_0 a)| \\
 &\geq |f(\Pi_0 a) - f(0)|^{1-\theta} \\
 &= |\mathcal{E}(\Psi(\Pi_0 a)) - \mathcal{E}(\Psi(a)) + \mathcal{E}(\Psi(a)) - \mathcal{E}(x_c)|^{1-\theta} \\
 &\geq |\mathcal{E}(x) - \mathcal{E}(x_c)|^{1-\theta} - C_3 \|\mathcal{M}(x)\|_{0,\alpha}^{2(1-\theta)}.
 \end{aligned}$$

Since $2(1-\theta) \geq 1$, (4.34) is established for some $C_2 > 0$ and for all $x \in \mathcal{W}_0$. \square

4.3. Proof of Theorem 1.1. The following lemma is first proved in [6] (Lemma 7.10 therein) when Σ is an n -dimensional self-shrinking embedded hypersurface in \mathbb{R}^{n+1} with polynomial growth. Since a branched conformal immersion is immersed away from finitely many points, the exact same proof holds for compact branched conformally immersed self-shrinkers in \mathbb{R}^m , $m \geq 3$. For the reader's convenience, we sketch the proof of Lemma 4.5 in the appendix. Note that the \mathcal{F} -functional (2.4) and the entropy (2.9) are also defined for branched immersions of compact surfaces.

Lemma 4.5. *Let $F : \Sigma \rightarrow \mathbb{R}^m$, $m \geq 3$, be a compact branched conformally immersed self-shrinking surface. Then the entropy λ defined in (2.9) is maximized at $(x_0, t_0) = (0, 1)$. That is,*

$$(4.45) \quad \lambda(F) = \frac{1}{4\pi} \int_{\Sigma} e^{-\frac{|F|^2}{4}} d\mu.$$

Note that if (F, τ) is a critical point of \mathcal{E} , then F is a branched conformally immersed self-shrinking surface. Conformality of F then implies $|DF|_{\tau}^2 d\mu_{\tau} = 2d\mu$, where $d\mu$ is the area element of the metric induced by F away from the branch points. Together with (4.3) and Lemma 4.5,

$$(4.46) \quad \mathcal{E}(F, \tau) = \int_{\mathbb{T}} e^{-\frac{|F|^2}{4}} d\mu = 4\pi\lambda(F).$$

Now we proceed to prove Theorem 1.1.

Proof. Assume the theorem is false. Then there is a sequence $\{F_n\} \in \mathfrak{X}_{\Lambda}$ with $\lambda(F_i) \neq \lambda(F_j)$ for all $i \neq j$. Let $g_n = F_n^*\langle \cdot, \cdot \rangle$ and let g_{τ_n} be the Riemannian metric on \mathbb{T} which is of the form (4.4) and is conformal to g_n . By Theorem 3.1, there is $F \in \mathfrak{X}_C$ and $\tau \in \mathbb{H}$ so that F_n converges smoothly to F and $\tau_n \rightarrow \tau$. Thus

$$\|(F_n, \tau_n) - (F, \tau)\|_{2,\alpha} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

From Proposition 4.2 and (4.46) and by setting $x_c = (F, \tau)$ in (4.34), we have $\lambda(F_i) = \lambda(F)$ for all i large enough, since $\mathcal{M}(F_n, \tau_n) = 0$ for all n . That leads to a contradiction. Thus the theorem is proved. \square

5. PIECEWISE LAGRANGIAN MEAN CURVATURE FLOWS

In this section, we extend the definition of the piecewise MCF in [6] to Lagrangian MCF for torus in \mathbb{R}^4 and construct a piecewise Lagrangian MCF for a Lagrangian immersed torus $F : \mathbb{T} \rightarrow \mathbb{R}^4$.

Definition 5.1. Let $F : L \rightarrow \mathbb{R}^4$ be a Lagrangian immersion, where L is a compact surface. A piecewise Lagrangian MCF with initial condition F is a finite collection of smooth Lagrangian MCFs

$$F_t^i : L \rightarrow \mathbb{R}^4$$

defined on $[t_i, t_{i+1}]$, $i = 0, 1, \dots, k-1$, where $0 = t_0 < t_1 < \dots < t_{k-1} < t_k < \infty$ so that:

- (1) $F_0^0 = F$,
- (2) $\mu(F_{t_{i+1}}^{i+1}) = \mu(F_{t_{i+1}}^i)$,
- (3) $\lambda(F_{t_{i+1}}^{i+1}) < \lambda(F_{t_{i+1}}^i)$,
- (4) there is $\delta > 0$ such that

$$(5.1) \quad \|F_{t_{i+1}}^i - F_{t_{i+1}}^{i+1}\|_{C^0} \leq \delta \sqrt{\mu(F_{t_{i+1}}^i)}$$

for $i = 0, 1, 2, \dots, k-2$.

Remark 3. Note that if $k = 1$, the piecewise MCF is just the usual smooth MCF. The above definition is interesting only if we can characterize the behaviour of the flow when $t \rightarrow t_k$.

Let $\{F_t : L \rightarrow \mathbb{R}^4\}$ be a smooth MCF defined on $[t_0, T_0)$, where $T_0 < \infty$ and L is a closed surface. Assume that a so-called type I singularity develops at T_0 , which means $\sup_{F_t(L)} \|A_t\| \rightarrow \infty$ as $t \rightarrow T_0$ and there is a positive constant C so that

$$(5.2) \quad \max_{F_t(L)} |A_t|^2 \leq \frac{C}{\sqrt{T_0 - t}}$$

for all $t < T_0$. Let $t_n \rightarrow T_0$ and $q_n \in F_{t_n}(L)$ where $\max_{F_{t_n}(L)} |A_{t_n}|$ is attained, and suppose $q_n \rightarrow q \in \mathbb{R}^4$. Consider the type I rescaling, which is the family of immersions $\tilde{F}(\cdot, s)$, where $-\log T_0 \leq s < \infty$ and

$$(5.3) \quad \tilde{F}(\cdot, s) = \frac{1}{\sqrt{(T_0 - t)}}(F_t(x) - q), \quad s(t) = -\log(T_0 - t).$$

For any sequence $s_j \rightarrow \infty$, a subsequence of $\{\tilde{F}(\cdot, s_j)\}$ converges locally smoothly to a self-shrinking immersion $\bar{F} : \Sigma \rightarrow \mathbb{R}^4$ ([14]). In this case, we say that *the type I singularity can be modelled by \bar{F}* . It is not known whether \bar{F} is unique: If we choose another sequence \tilde{s}_k , $\{\tilde{F}(\cdot, \tilde{s}_k)\}$ might converge to a different self-shrinker.

Now we prove Theorem 1.3.

Proof. Let $F : \mathbb{T} \rightarrow \mathbb{R}^4$ be a Lagrangian immersion. By [28], there is a unique smooth Lagrangian MCF $\{F_t\}$ which is defined on a maximal time interval $[0, T_0)$, where $T_0 < \infty$ as \mathbb{T} is compact.

If the singularity at T_0 is not a type I singularity that can be modelled by a compact self-shrinker with area no larger than Λ , then we set $k = 0$ and no perturbation is performed.

Otherwise, the singularity at T_0 is of type I and it can be modelled by a compact self-shrinker with area no larger than Λ . In this case, the inequality (5.2) is satisfied at a point $q \in \mathbb{R}^4$ at time T_0 for some positive constant C and for all $t \in [0, T_0)$, and there is a sequence $s_j \rightarrow \infty$ such that $\tilde{F}(\cdot, s_j)$ as in (5.3) converges locally smoothly to a compact self-shrinker \bar{F} with area no bigger than Λ . To be precise about the convergence, we recall that Lemma 3.3, Corollary 3.2 and Proposition 2.3 in [14] hold for any codimension, and they guarantee that all $\tilde{F}(\cdot, s_j)$ touch a fixed bounded region, the areas inside a ball $B(R)$ are bounded by $C(R)$ and the second fundamental forms and their derivatives of any order are bounded. Therefore, all the conditions in Theorem 1.3 in [3] are satisfied for the sequence $\{\tilde{F}(\cdot, s_j)\}$, and the theorem asserts: by passing to a subsequence if necessary, there is a surface Σ and an immersion $\bar{F} : \Sigma \rightarrow \mathbb{R}^4$ and a sequence of diffeomorphisms

$$\varphi_j : U^j \rightarrow \tilde{F}(\cdot, s_j)^{-1}(B_j) \subset \mathbb{T},$$

where B_j is the ball of radius j in \mathbb{R}^4 centered at the origin, $U_j \subset \Sigma$ are open sets with $U_j \subset\subset U_{j+1}$ and $\Sigma = \bigcup_j U_j$, such that

$$\|\tilde{F}(\cdot, s_j) \circ \varphi_j - \bar{F}\|_{C^0(U_j)} \rightarrow 0$$

and $\tilde{F}(\cdot, s_j) \circ \varphi$ converges to \bar{F} locally smoothly. In our situation, we have assumed that Σ is compact (as we are dealing with singularity that can be modelled by compact shrinkers). Hence $\Sigma = U_k$ for all k large and thus φ_k are diffeomorphisms from Σ to \mathbb{T} , since the torus is connected. To simplify notations, we write $\Sigma = \mathbb{T}$. The diffeomorphisms $\varphi_j : \mathbb{T} \rightarrow \mathbb{T}$ have the property that

$$(5.4) \quad \|\tilde{F}(\cdot, s_j) \circ \varphi_j - \bar{F}\|_{C^k(\mathbb{T})} \rightarrow 0$$

for all $k = 0, 1, 2, \dots$. Since each $\{F_t\}$ is Lagrangian, the sequence of blowups $\tilde{F}(\cdot, s_j)$ are also Lagrangian for all j . The above convergence implies that \bar{F} is Lagrangian, hence, $\bar{F} \in \mathfrak{X}_\Lambda$.

Since the entropy λ (2.9) is translation and scaling invariant,

$$(5.5) \quad \lambda(\tilde{F}(\cdot, s(t))) = \lambda(F_t).$$

Furthermore, by the definition of \mathcal{F}_{x_0, t_0} in (2.4), we see

$$(5.6) \quad \lambda(\tilde{F}(\cdot, s_j) \circ \varphi_j) = \lambda(\tilde{F}(\cdot, s_j)).$$

Since $\mathcal{F}_{0,1}$ (see (2.4)) is continuous with respect to the C^1 -topology, there is a sequence d_j of positive numbers so that $d_j \rightarrow 0$ as $j \rightarrow \infty$ and

$$\mathcal{F}_{0,1}(\tilde{F}(\cdot, s_j) \circ \varphi_j) \geq \mathcal{F}_{0,1}(\bar{F}) - d_j.$$

By definition of λ and Lemma 4.5, since \bar{F} is a self-shrinker, from the above we have

$$(5.7) \quad \lambda(\tilde{F}(\cdot, s_j) \circ \varphi_j) \geq \lambda(\bar{F}) - d_j.$$

As λ is non-increasing along the MCF, $\lambda(\tilde{F}(\cdot, s_j))$ is non-increasing in j by (5.5). Together with (5.6) and (5.7), we conclude

$$\lambda(\tilde{F}(\cdot, s_j)) \geq \lim_{j \rightarrow \infty} \lambda(\tilde{F}(\cdot, s_j)) \geq \lambda(\overline{F}).$$

Fix $\delta > 0$. Let

$$\delta_1 = \frac{\delta \sqrt{\mu(\overline{F})}}{6}, \quad \delta_2 = \min \left\{ \frac{1}{2}, \frac{\delta_1}{\|\overline{F}\|_{C^0} + \delta_1} \right\}.$$

Using (5.4), for all $k \geq 1$, there is j_0 so that

$$(5.8) \quad \|\tilde{F}(\cdot, s_{j_0}) \circ \varphi_{j_0} - \overline{F}\|_{C^k} < \delta_1,$$

and

$$(5.9) \quad \left| \frac{\mu(\tilde{F}(\cdot, s_{j_0}))}{\mu(\overline{F})} - 1 \right| \leq \delta_2.$$

By Theorem 2.1, \overline{F} is Lagrangian \mathcal{F} -unstable. Then by Theorem 2.2, there is a Lagrangian immersion $\widehat{F} : \mathbb{T} \rightarrow \mathbb{R}^4$ which satisfies

$$(5.10) \quad \|\widehat{F} - \overline{F}\|_{C^2} < \delta_1,$$

$$(5.11) \quad \left| \frac{\mu(\overline{F})}{\mu(\widehat{F})} - 1 \right| \leq \delta_2$$

and

$$(5.12) \quad \lambda(\widehat{F}) < \lambda(\overline{F}).$$

Now we define the first part of the piecewise Lagrangian MCF:

- (i) The first piece of Lagrangian MCF is just $F_t^0 := F_t$, where $t \in [0, t_1]$ and $t_1 < T_0$ is such that $s(t_1) = s_{j_0}$.
- (ii) Define the first perturbation $F_{t_1}^1$ at time t_1 as

$$(5.13) \quad F_{t_1}^1 = \sqrt{T_0 - t_1}(\kappa \widehat{F}) \circ \varphi_{j_0}^{-1} + q.$$

where the dilation factor

$$\kappa = \sqrt{\frac{\mu(\tilde{F}(\cdot, s_{j_0}))}{\mu(\widehat{F})}}.$$

The constant κ is chosen so that

$$(5.14) \quad \mu(\kappa \widehat{F}) = \mu(\tilde{F}(\cdot, s_{j_0})).$$

We check now that (2)-(4) in definition 5.1 are satisfied with $i = 0$. First note that (2) follows from (5.14) and the definition of $F_{t_1}^0$ and $F_{t_1}^1$. To prove (3), since the entropy (2.9) is scaling and translation invariant, using $\lambda(\overline{F}) > \lambda(\widehat{F})$ we obtain

$$\lambda(F_{t_1}^0) = \lambda(\tilde{F}(\cdot, s_{j_0})) \geq \lambda(\overline{F}) > \lambda(\widehat{F}) = \lambda(F_{t_1}^1).$$

Thus (3) is also shown. Lastly, we show that (5.1) is satisfied with $i = 0$. From (5.3) and (5.13), we have

$$\|F_{t_1}^0 - F_{t_1}^1\|_{C^0} = \sqrt{T_0 - t_1} \|\tilde{F}(\cdot, s_{j_0}) \circ \varphi_{j_0} - \kappa \hat{F}\|_{C^0}.$$

Note that (5.9) and (5.11) imply

$$(5.15) \quad |\kappa - 1| \leq B.$$

Together with (5.10), (5.8), the definition of δ_2 , we have

$$\begin{aligned} \|\tilde{F}(\cdot, s_{j_0}) \circ \varphi_{j_0} - \kappa \hat{F}\|_{C^0} &\leq \|\tilde{F}(\cdot, s_{j_0}) \circ \varphi_{j_0} - \bar{F}\|_{C^0} + \|\bar{F} - \hat{F}\|_{C^0} + \|(1 - \kappa)\hat{F}\|_{C^0} \\ &\leq 2\delta_1 + \delta_2(\delta_1 + \|\tilde{F}\|_{C^0}) \\ &\leq 3\delta_1, \end{aligned}$$

where we used the simple estimate

$$\|\hat{F}\|_{C^0} \leq \|\hat{F} - \bar{F}\|_{C^0} + \|\bar{F}\|_{C^0}.$$

Thus we have

$$\begin{aligned} \|F_{t_1}^0 - F_{t_1}^1\|_{C^0} &< 3\delta_1 \sqrt{T_0 - t_1} \\ &= 3\delta_1 \sqrt{\frac{\mu(F_{t_1}^0)}{\mu(\tilde{F}(\cdot, s_{j_0}))}} \\ &\leq \frac{1}{2}\delta \sqrt{\mu(F_{t_1}^0)} \sqrt{\frac{\mu(\bar{F})}{\mu(\tilde{F}(\cdot, s_{j_0}))}} \\ &\leq \delta \sqrt{\mu(F_{t_1}^0)}, \end{aligned}$$

where in the last step we used $\delta_2 \leq \frac{1}{2}$. Thus (5.1) is shown and this finishes the construction of the first piece of the piecewise Lagrangian MCF.

Using $F_{t_1}^1$ as initial condition, there is another family $\{F_t : t \in [t_1, T_1]\}$ of smooth Lagrangian MCF with $F_{t_1} = F_{t_1}^1$. Again, if the condition in Theorem 1.3 is satisfied at the singular time T_1 (that is, the singularity at T_1 is not of type I which can be modelled by a compact self-shrinker of area $\leq \Lambda$), then we set $k = 1$, $t_2 = T_1$ and we are done. If not, we carry out exactly the same procedure as above. Thus we have a Lagrangian self-shrinking torus $\bar{F}_1 \in \mathfrak{X}_\Lambda$, some time $t_2 < T_1$ and another Lagrangian immersion $F_{t_2}^2$ so that

$$\begin{aligned} \lambda(F_{t_2}^1) &\geq \lambda(\bar{F}_1) > \lambda(F_{t_2}^2), \\ \mu(F_{t_2}^1) &= \mu(F_{t_2}^2) \end{aligned}$$

and

$$\|F_{t_2}^1 - F_{t_2}^2\|_{C^0} < \delta \sqrt{\mu(F_{t_2}^1)}.$$

Then, again, we apply the smooth Lagrangian MCF to $F_{t_2}^2$. Note that the above procedure must stop: Indeed, by Theorem 1.1, the image of $\lambda : \mathfrak{X}_\Lambda \rightarrow \mathbb{R}$ is finite. Moreover, from the above construction, each perturbation is chosen so that the entropy value is strictly less than one of the element in $\lambda(\mathfrak{X}_\Lambda)$. Since λ is non-increasing along the usual MCF,

the above procedure must terminate after k steps for some $k \leq |\lambda(\mathfrak{X}_\Lambda)|$. This implies that at t_k , the piecewise Lagrangian MCF do not encounter a type I singularity which can be modelled by a compact self-shrinker with area less than or equals to Λ .

To prove the last statement of Theorem 1.3, recall that the Maslov class of a Lagrangian immersion is given by $2[H] \in H^1(\mathbb{T}, \mathbb{Z})$, where H is the mean curvature form and $[H]$ is an integral class as

$$(5.16) \quad H = d\theta,$$

where $\theta : \mathbb{T} \rightarrow \mathbb{S}^1$ is the Lagrangian angle of the immersion $F : \mathbb{T} \rightarrow \mathbb{R}^4$ [13]. When $\{F_t\}$ is a smooth Lagrangian MCF, $[H_t]$ is invariant as $[H_t]$ is an integral class and H_t is smooth in t . This fact can also be checked using the evolution of H under the Lagrangian MCF, see Theorem 2.9 in [28]. From (5.16) it is also clear that the Maslov class is invariant under translation and scaling of the immersion. Thus when there is a type I singularity and $\overline{F} : \mathbb{T} \rightarrow \mathbb{R}^4$ is a compact Lagrangian self-shrinker which models the singularity, then $[H_{\overline{F}}] = [H_t]$. Lastly, we recall that in Theorem 2.2 the perturbation \widehat{F} is defined using a closed 1-form on \mathbb{T} . Hence we also have $[H_{\overline{F}}] = [H_{F_{t_1}^1}]$. Thus the Maslov class is preserved when we perturb the Lagrangian immersion in constructing the piecewise Lagrangian MCF. This completes the proof of Theorem 1.3. \square

5.1. Generalization to Lagrangian immersion of higher genus surfaces. Theorem 1.3 can be extended to genus $g > 1$ if we impose further assumptions on the singularity. Let $c_1, c_2 > 0$ and consider the set $\mathfrak{X}_{g, c_1, c_2}^{\text{imm}}$ of all Lagrangian self-shrinking immersions $\overline{F} : \Sigma_g \rightarrow \mathbb{R}^4$ with area $\leq c_1$ and the second fundamental form satisfying $\max_{\overline{F}(\Sigma_g)} |A| \leq c_2$, where Σ_g is a closed orientable surface of genus g with $g > 1$. Using (2.2), there are constants $C(k, c_1, c_2) > 0$ that depend on c_1, c_2, k , such that

$$\max_{\overline{F}(\Sigma_g)} |\nabla^k A| \leq C(k, c_1, c_2)$$

for all $\overline{F} \in \mathfrak{X}_{g, c_1, c_2}^{\text{imm}}$. Thus we can apply Theorem 1.3 in [3] to conclude that $\mathfrak{X}_{g, c_1, c_2}^{\text{imm}}$ is compact in the C^2 -topology, in particular, all sequential limits are unbranched. Unbranchedness of any limiting surface guarantees existence of nearby Lagrangian immersions by the Lagrangian neighbourhood theorem. By Theorem 2.1 and Theorem 2.2 again, the Lagrangian self-shrinkers in $\mathfrak{X}_{g, c_1, c_2}^{\text{imm}}$ are Lagrangian entropy unstable. It follows that all $\overline{F} \in \mathfrak{X}_{g, c_1, c_2}^{\text{imm}}$ are Lagrangian entropy unstable. With these facts, the proof of the following proposition is identical to that of Corollary 8.4 in [6] and is omitted here.

Proposition 5.1. *Let $\delta > 0$. Then there is a positive constant c depending only on δ such that for any Lagrangian self-shrinker $\overline{F} \in \mathfrak{X}_{g, c_1, c_2}^{\text{imm}}$, there is a Lagrangian immersion $\widehat{F} : \Sigma_g \rightarrow \mathbb{R}^4$ so that $\|\widehat{F} - \overline{F}\|_{C^0} < \delta \sqrt{\mu(\overline{F})}$ and $\lambda(\widehat{F}) < \lambda(\overline{F}) - c$.*

Remark 4. For genus > 1 , without assuming uniform boundedness of the second fundamental forms, the compactness result in Theorem 3.1 is not enough to conclude Proposition 5.1 due to the assumption on the conformal structures there.

Using Proposition 5.1, we can define a piecewise Lagrangian MCF for a Lagrangian immersion $F : \Sigma_g \rightarrow \mathbb{R}^4$, as we did in the genus 1 case. After each perturbation, the entropy decreases by a fixed amount $c > 0$ (Note that this c might depend on δ). Since the entropy is always a positive number, we conclude that the process must terminate in finite time and we have the following

Theorem 5.1. *Let $F : \Sigma_g \rightarrow \mathbb{R}^4$ be a Lagrangian immersion and $\delta > 0$ be given. Then there exists a piecewise Lagrangian MCF $\{F_t^i : i = 0, 1, \dots, k-1\}$ with initial condition F , such that the singularity at time t_k is not a type I singularity which can be modelled by a self-shrinker in $\mathfrak{X}_{g, c_1, c_2}^{\text{imm}}$. Moreover, we have the estimates $\|F_{t_i}^i - F_{t_i}^{i+1}\|_{C^0} < \delta \sqrt{\mu(F_{t_i}^i)}$ and the Maslov class of each immersion is invariant along the flow.*

6. APPENDIX

6.1. Proof of Lemma 4.5. Let Σ be a compact surface without boundary and let $F : \Sigma \rightarrow \mathbb{R}^4$ be a branched conformal self-shrinker. Define the operator \mathcal{L}_s by

$$(6.1) \quad \mathcal{L}_s u = \Delta u - \frac{1}{2t_s} \langle (x - x_s)^\top, \nabla u \rangle = e^{\frac{|x-x_s|^2}{4t_s}} \operatorname{div} \left(e^{-\frac{|x-x_s|^2}{4t_s}} \nabla u \right).$$

Here $(x_s, t_s) \in \mathbb{R}^4 \times \mathbb{R}_{>0}$, ∇ , div and Δ are taken with respect to the pullback metric $F^* \langle \cdot, \cdot \rangle$ and u, v are functions on \mathbb{R}^4 . Note that \mathcal{L}_s is defined away from the set of branch points B . As in [6], we use the square bracket $[\cdot]_s$ to denote

$$(6.2) \quad [f]_s = \frac{1}{4\pi t_s} \int_{\Sigma} f e^{-\frac{|x-x_s|^2}{4t_s}} d\mu$$

Lemma 6.1. *We have*

$$(6.3) \quad [u\mathcal{L}_s v]_s = -[\langle \nabla u, \nabla v \rangle]_s.$$

Proof. Let $B = \{x_1, \dots, x_n\}$. Let $\epsilon > 0$ be small and $B_i(\epsilon)$ be an ϵ -ball in Σ with center x_i , so that $B_i(\epsilon) \cap B_j(\epsilon) = \emptyset$ if $i \neq j$. Then

$$(6.4) \quad \begin{aligned} [u\mathcal{L}v]_s &= \frac{1}{4\pi t_s} \int_{\Sigma} u \operatorname{div} \left(e^{-\frac{|x-x_s|^2}{4t_s}} \nabla v \right) d\mu \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi t_s} \int_{\Sigma \setminus \bigcup B_i(\epsilon)} u \operatorname{div} \left(e^{-\frac{|x-x_s|^2}{4t_s}} \nabla v \right) d\mu \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi t_s} \left(\sum_i \int_{\partial B_i(\epsilon)} u \langle \nabla v, n_i \rangle e^{-\frac{|x-x_s|^2}{4t_s}} dl - \int_{\Sigma \setminus \bigcup B_i(\epsilon)} \langle \nabla u, \nabla v \rangle e^{-\frac{|x-x_s|^2}{4t_s}} d\mu \right) \\ &= -[\langle \nabla u, \nabla v \rangle]_s \end{aligned}$$

where n_i is the unit outward normal along $\partial B_i(\epsilon)$. □

In particular, we have

$$(6.5) \quad [u\mathcal{L}_s v]_s = -[\langle \nabla u, \nabla v \rangle]_s = [v\mathcal{L}_s u]_s.$$

Using (6.5), exactly the same argument in [6], pp. 786-788, shows that for all $y \in \mathbb{R}^4$ and $a \in \mathbb{R}$ if we set $(x_s, t_s) = (sy, 1 + as^2)$ and $g(s) = \mathcal{F}_{x_s, t_s}(F)$ then $g'(s) \leq 0$ for all $s > 0$ with $1 + as^2 > 0$. Thus $\mathcal{F}_{y, t}(F) \leq \mathcal{F}_{0, 1}(F)$ for all $(y, t) \in \mathbb{R}^4 \times \mathbb{R}_{>0}$ and thus Lemma 4.5 is proved.

6.2. Analyticity of \mathcal{E} and \mathcal{M} . Next we show that both \mathcal{E} and \mathcal{M} defined in (4.3) and (4.5) are analytic. For the definition of continuous symmetric n -linear form and analytic function between Banach spaces, please refer to Chapter 4 in [9]. First we have

Lemma 6.2. *Let X, Y, Z be Banach spaces, \mathcal{U}, \mathcal{V} are open in X, Y respectively, and $f : \mathcal{U} \rightarrow \mathbb{R}, g : \mathcal{V} \rightarrow Z$ are analytic at $x_0 \in \mathcal{U}, y_0 \in \mathcal{V}$ respectively. Then the function*

$$h : \mathcal{U} \times \mathcal{V} \rightarrow Z, \quad h(x, y) = f(x)g(y)$$

is analytic at (x_0, y_0) .

Proof. Since f, g are analytic at x_0, y_0 respectively, then

$$(6.6) \quad f(x_0 + h) = f(x_0) + \sum_{i=1}^{\infty} A_i(h^i), \quad g(y_0 + k) = g(y_0) + \sum_{j=1}^{\infty} B_j(k^j)$$

for all $\|h\|_X < \epsilon_1$ and $\|k\|_Y < \epsilon_2$, and A_i, B_j are continuous multi-linear forms such that

$$(6.7) \quad \sum_{i=1}^{\infty} \|A_i\| \epsilon_1^i < +\infty \quad \text{and} \quad \sum_{j=1}^{\infty} \|B_j\| \epsilon_2^j < +\infty.$$

The absolute convergence of (6.6) implies that

$$(6.8) \quad h(x_0 + h, y_0 + k) = f(x_0)g(y_0) + \sum_{n=1}^{\infty} C_n(h, k),$$

for all (h, k) such that $\|h\|_X < \epsilon_1, \|k\|_Y < \epsilon_2$, where

$$(6.9) \quad C_n(h, k) = \sum_{i=0}^n A_i(h^i) B_{n-i}(k^{n-i}).$$

Let $\epsilon = \frac{1}{2} \min\{\epsilon_1, \epsilon_2\}$. Then by definition of $\|C_n\|$ and ϵ , one has

$$\begin{aligned} \|C_n\| \epsilon^n &= \sup_{\|h\| + \|k\| = \epsilon} \|C_n(h, k)\|_Z \\ &\leq \sum_{i=0}^n (\|A_i\| \epsilon_1^i) (\|B_{n-i}\| \epsilon_2^{n-i}). \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \|C_n\| \epsilon^n \leq \left(\sum_{i=1}^{\infty} \|A_i\| \epsilon_1^i \right) \left(\sum_{j=1}^{\infty} \|B_j\| \epsilon_2^j \right) < +\infty$$

by (6.7). Hence h is also analytic at (x_0, y_0) . □

Proposition 6.1. *The mapping $\mathcal{E} : \mathcal{U} \rightarrow \mathbb{R}$ in (4.3) is analytic.*

Proof. Using (4.4), we have

$$(6.10) \quad 2\mathcal{E}(u, \tau) = (\tau_1^2/\tau_2 + \tau_2) L_{11}(u) - (2\tau_1/\tau_2)L_{12}(u) + L_{22}(u),$$

where

$$(6.11) \quad L_{ij}(u) = \int_{\mathbb{T}} D_i u \cdot D_j u e^{-\frac{|u|^2}{4}} dx dy.$$

Since $\tau \mapsto (\tau_1^2/\tau_2) + \tau_2$ and $\tau \mapsto \tau_1/\tau_2$ are analytic, by Lemma 6.2, it suffices to check $L_{ij} : C^{2,\alpha} \rightarrow \mathbb{R}$ is analytic. But this is obvious, using the power series expansion of $e^{-\frac{|u|^2}{4}}$. \square

Proposition 6.2. *The mapping $\mathcal{M} : \mathcal{U} \mapsto \mathcal{C}^{0,\alpha}$ in (4.5) is analytic.*

Proof. It suffices to show that both components in (4.7) are analytic. The second component $(u, \tau) \mapsto \nabla \mathcal{E}_\tau^u$ is analytic since \mathcal{E} is analytic by Proposition 6.1, here we recall that $\nabla \mathcal{E}_\tau^u$ is the gradient of $\mathcal{E}(u, \tau)$ at τ . Note that the first component can be written as

$$(6.12) \quad (u, \tau) \mapsto -g_\tau^{ij} \left(D_{ij} u - (u \cdot D_j u) D_i u + \frac{1}{4} (D_i u \cdot D_j u) u \right)$$

Since $\tau \mapsto g_\tau^{ij}$ is analytic, the mapping in (6.12) is also analytic by Lemma 6.2. \square

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